Priestley duality for (modal) N4-lattices*

Ramon Jansana and Umberto Rivieccio

[1] Department of Logic, History and Philosophy of Science
University of Barcelona
Spain
Jansana@ub.edu
[2] School of Computer Science
University of Birmingham
United Kingdom
U.Rivieccio@cs.bham.ac.uk

N4-lattices are the algebraic semantics of *paraconsistent Nelson logic*, which was introduced in [1] as an inconsistency-tolerant counterpart of the better-known logic of Nelson [7, 13]. Paraconsistent Nelson logic combines interesting features of intuitionistic, classical and many-valued logics (e.g., Belnap-Dunn four-valued logic); recent work has shown that it can also be seen as one member of the wide family of substructural logics [15].

The work we present here is a contribution towards a better topological understanding of the algebraic counterpart of paraconsistent Nelson logic, namely a variety of involutive lattices called *N4-lattices* in [8]. A Priestley-style duality for these algebras has already been introduced by Odintsov [10]. The main difference between his approach and ours is that we only rely on Esakia duality for Heyting algebras [4], whereas [10] uses both Esakia duality and the duality for De Morgan algebras [2, 3]; as a consequence, the description of dual spaces that we obtain is, in our opinion, much simpler. Moreover, [10] only deals with N4-lattices whose lattice reduct is bounded, whereas we show that our treatment extends to the non-bounded case as well. We also consider N4-lattices expanded with a monotone modal operator, which have been recently introduced in the algebraic investigation of modal expansions of Belnap-Dunn logic [12, 11, 14]. Building on duality theory for distributive lattices with modal operators [5, 6], we introduce a duality for these *modal N4-lattices*, which can moreover be employed to provide a neighborhood semantics for the logic of [14].

Definition 1. *An* N4-lattice *[9, Definition 2.3] is an algebra* $\mathbf{B} = \langle B, \wedge, \vee, \to, \sim \rangle$ *such that:*

(i) the reduct $\langle B, \wedge, \vee, \sim \rangle$ *is a De Morgan lattice, i.e., a distributive lattice (with order* $\leq$*) equipped with a unary operation* $\sim \colon B \to B$ *(usually called* negation*) such that* $\sim\sim a = a$ *and* $\sim(a \vee b) = \sim a \wedge \sim b$*,*

(ii) the relation $\preceq$ *defined as* $a \preceq b$ *iff* $a \to b = (a \to b) \to (a \to b)$*, is a pre-ordering (i.e., reflexive and transitive),*

(iii) the relation $\equiv$ *defined as* $a \equiv b$ *iff* $a \preceq b$ *and* $b \preceq a$ *is a congruence w.r.t.* $\wedge, \vee, \to$ *and the quotient algebra* $\mathbf{B}_{\bowtie} = \langle B, \wedge, \vee, \to \rangle / \equiv$ *is a Brouwerian lattice*[1]*,*

(iv) $\sim(a \to b) \equiv a \wedge \sim b$ *for all* $a, b \in B$*,*

(v) $a \leq b$ *iff* $a \preceq b$ *and* $\sim b \preceq \sim a$ *for all* $a, b \in B$*.*

---

*Submission to the conference on *Topology, Algebra and Categories in Logic (TACL 2013)*, to be held in Nashville, Tennessee (USA), on July 28 - August 1, 2013. The corresponding author is the second one.

[1] A *Brouwerian lattice* is a the 0-free subreduct of a Heyting algebra. Brouwerian lattices are also known in the literature as *generalized Heyting algebras*, *Brouwerian algebras*, *implicative lattices* [8] or *relatively pseudo-complemented lattices* [13]. Some authors call "Brouwerian lattices" structures that are (lattice-theoretic) dual to ours.

We say that $\mathbf{B}$ *is* bounded *if its lattice reduct is bounded (in which case* $\mathbf{B}_{\bowtie}$ *is a Heyting algebra).*

It is known [8] that N4-lattices form a variety (hence, bounded N4-lattices are also a variety), which is the equivalent algebraic semantics of paraconsistent Nelson logic.

Definition 1 may seem rather obscure; fortunately a more insightful description is available for these algebras thanks to the so-called *twist-structure* construction [9, Definition 2.1], which allows us to view any N4-lattice as a special power of a Brouwerian lattice.

Let $\mathbf{A} = \langle A, \wedge, \vee, \to, 1 \rangle$ be a Brouwerian lattice. Consider the algebra $\mathbf{A}^{\bowtie} = \langle A \times A, \wedge, \vee, \to, \sim \rangle$ with operations defined as follows: for all $\langle a, b \rangle, \langle c, d \rangle \in A \times A$,

$$
\begin{aligned}
\langle a,b\rangle \wedge \langle c,d\rangle &:= \langle a \wedge c, b \vee d\rangle \\
\langle a,b\rangle \vee \langle c,d\rangle &:= \langle a \vee c, b \wedge d\rangle \\
\langle a,b\rangle \to \langle c,d\rangle &:= \langle a \to c, a \wedge d\rangle \\
\sim\langle a,b\rangle &:= \langle b,a\rangle.
\end{aligned}
$$

It is easy to check that $\mathbf{A}^{\bowtie}$ is an N4-lattice. If $\mathbf{A}$ has a minimum element 0, then letting $\top := \langle 1, 0 \rangle$ and $\bot := \langle 0, 1 \rangle$ we obtain a bounded N4-lattice. However, not all N4-lattices can be obtained in this way. To achieve this, we need the following refinement. Let $D(\mathbf{A}) := \{a \vee (a \to b) : a, b \in A\}$ denote the set of *dense elements* of the Brouwerian lattice $\mathbf{A}$. Consider a lattice filter $\nabla \subseteq A$ such that $D(\mathbf{A}) \subseteq \nabla$ and an arbitrary (non-empty) lattice ideal $\Delta \subseteq A$. Then the set $B := \{\langle a, b \rangle \in A \times A : \ a \vee b \in \nabla, \ a \wedge b \in \Delta\}$ is closed under the operations $\wedge, \vee, \to, \sim$ of $\mathbf{A}^{\bowtie}$ and thus $\langle B, \wedge, \vee, \to, \sim \rangle$ is an N4-lattice. Following [9], we denote this algebra by $Tw(\mathbf{A}, \nabla, \Delta)$ and call it a *twist-structure over* $\mathbf{A}$.

Odintsov [9, Corollary 3.2] proved that *any N4-lattice* $\mathbf{B}$ *is isomorphic to the algebra* $Tw(\mathbf{B}_{\bowtie}, \nabla, \Delta)$, where $\mathbf{B}_{\bowtie}$ is the Brouwerian lattice of Definition 1 (iii) and $\nabla(\mathbf{B}), \Delta(\mathbf{B})$ are defined as follows:

$$
\nabla(\mathbf{B}) := \{[a \vee \sim a] : a \in B\} \qquad \Delta(\mathbf{B}) := \{[a \wedge \sim a] : a \in B\},
$$

where $[b]$ is the equivalence class of $b \in B$ modulo the relation $\equiv$ also introduced in Definition 1 (iii). The isomorphism $j_{\mathbf{B}} \colon \mathbf{B} \cong Tw(\mathbf{B}_{\bowtie}, \nabla, \Delta)$ is given by the map $j_{\mathbf{B}} \colon B \to B/{\equiv} \times B/{\equiv}$ defined as $j_{\mathbf{B}}(a) := \langle [a], [\sim a] \rangle$ for all $a \in B$.

The one-to-one correspondence between N4-lattices and triples of the form $(\mathbf{A}, \nabla, \Delta)$ established above can be extended to a categorial equivalence between (1) the category $\mathsf{N4}$, whose objects are (bounded) N4-lattices and whose morphisms are algebraic (bounded) N4-lattice homomorphisms, and (2) the category Twist whose objects are triples $(\mathbf{A}, \nabla, \Delta)$ with $\mathbf{A}$ a (bounded) Brouwerian lattice, $\nabla \subseteq A$ a lattice filter containing the dense elements $D(\mathbf{A})$, and $\Delta \subseteq A$ an ideal, and whose morphisms are (bounded) Brouwerian lattice homomorphisms $h \colon \mathbf{A}_1 \to \mathbf{A}_2$ such that $h[\nabla_1] \subseteq \nabla_2$ and $h[\Delta_1] \subseteq \Delta_2$. This equivalence easily follows from Odintsov's results but, to our knowledge, we are the first ones to prove it formally. Notice that an object of Twist is for us a triple $(\mathbf{A}, \nabla, \Delta)$, not the product algebra $Tw(\mathbf{A}, \nabla, \Delta)$ introduced before.

Using this result, together with Esakia duality for Heyting algebras, we can obtain a topological duality for bounded N4-lattices via their twist-structure representation. Recall that Esakia duality is a dual equivalence between the category of Heyting algebras (with algebraic homomorphisms as morphisms) and the category of Esakia spaces (with Esakia functions as morphisms), defined as follows. An *Esakia space* is a Priestley space $= \langle X, \tau, \leq \rangle$ such that the down-set $\downarrow U$ of every clopen $U \subseteq X$ is clopen. An *Esakia function* is a map $f : X \to Y$ between Esakia spaces X, Y which is Priestley-continuous and such that $\uparrow_Y f(x) \subseteq f[\uparrow_X x]$ for every $x \in X$. The functors that establish Esakia duality are defined in the same way as in Priestley duality.

It follows from Priestley duality that the lattice filters of a Heyting algebra are in one-to-one correspondence with closed up-sets of its dual Esakia space, and, similarly, lattice ideals correspond to open up-sets of the dual space. This allows us to establish a correspondence between a triple $(\mathbf{A}, \nabla, \Delta) \in \mathsf{Twist}$ and a topological structure $\langle X, \leq, \tau, C, O \rangle$, which we call a *Nelson-Esakia space*, such that:

- $\langle X, \leq, \tau \rangle$ is an Esakia space,
- $C \subseteq X$ is a closed set such that $C \subseteq \max(X)$,
- $O \subseteq X$ is an open up-set.

Here $\max(X)$ denotes the set of maximal elements of the poset $\langle X, \leq \rangle$, and we note that the condition $C \subseteq \max(X)$ is meant to reflect the fact that the filter ∇ contains the dense elements.

A category $\mathsf{NE\text{-}Sp}$ of Nelson-Esakia spaces is obtained by adopting the following notion of morphism. A morphism between Nelson-Esakia spaces $\langle X_1, \leq_1, \tau_1, C_1, O_1 \rangle$ and $\langle X_2, \leq_2, \tau_2, C_2, O_2 \rangle$ is an Esakia function $f : X_1 \to X_2$ that satisfies $f[C_1] \subseteq C_2$ and $f^{-1}[O_2] \subseteq O_1$.

A dual equivalence between the category Twist of twist-structures over Heyting algebras and the category $\mathsf{NE\text{-}Sp}$ of Nelson-Esakia spaces can be established using the same functors involved in Priestley (and Esakia) duality. The only non-trivial bit is to take care of the additional structure which is given, on the algebraic side, by ∇ and Δ, and on the topological side by the sets C, O defined above. This can be done as follows.

To a twist-structure $(\mathbf{A}, \nabla, \Delta)$ we associate the Nelson-Esakia space $\langle X(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq, C_{\mathbf{A}}, O_{\mathbf{A}} \rangle$, where $\langle X(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq \rangle$ is the usual dual Esakia space of $\mathbf{A}$ and

$$C_{\mathbf{A}} := \{P \in X(\mathbf{A}) : \nabla \subseteq P\} \qquad O_{\mathbf{A}} := \{P \in X(\mathbf{A}) : P \cap \Delta \neq \emptyset\}.$$

Conversely, to a Nelson-Esakia space $\langle X, \leq, \tau, C, O \rangle$ we associate the twist-structure $\langle A(X), \nabla_C, \Delta_O \rangle$, where $A(X)$ is the algebra of clopen up-sets of X (which carries, by Esakia duality, the structure of a Heyting algebra) and

$$\nabla_C := \{U \in A(X) : C \subseteq U\} \qquad \Delta_O := \{U \in A(X) : U \subseteq O\}.$$

Joining the results stated so far we obtain the equivalences displayed below.

$$\begin{tikzcd} \mathsf{N4} \arrow[r, bend left, "T"] & \mathsf{Twist} \arrow[l, bend left, "N"] \arrow[r, bend left, "X"] & \mathsf{NE\text{-}Sp}^{op} \arrow[l, bend left, "A"] \end{tikzcd}$$

Given an N4-lattice $\mathbf{B}$, we let $T(\mathbf{B}) := (\mathbf{B}_{\bowtie}, \nabla(\mathbf{B}), \Delta(\mathbf{B}))$. If $f \colon \mathbf{B_1} \to \mathbf{B_2}$ is an $\mathsf{N4}$-morphism, we define $T(f) \colon (\mathbf{B_1})_{\bowtie} \to (\mathbf{B_2})_{\bowtie}$ as $T(f)([a]_{\equiv_1}) := [f(a)]_{\equiv_2}$, where $[a]_{\equiv_1}$ is the equivalence class of $a \in B_1$ modulo the relation introduced in Definition 1 (iii) and likewise $[b]_{\equiv_2} \in B_2/{\equiv_2}$ for all $b \in B_2$. Conversely, for $(\mathbf{A}, \nabla, \Delta) \in \mathsf{Twist}$, we let $N(\mathbf{A}, \nabla, \Delta) := Tw(\mathbf{A}, \nabla, \Delta)$. For a morphism $h \colon A_1 \to A_2$ between twist-structures $(\mathbf{A}_1, \nabla_1, \Delta_1)$, and $(\mathbf{A}_2, \nabla_2, \Delta_2)$, we define the map $N(h) \colon N(\mathbf{A}_1, \nabla_1, \Delta_1) \to N(\mathbf{A}_2, \nabla_2, \Delta_2)$, as $N(h)\langle a, b \rangle := \langle h(a), h(b) \rangle$ for all $a, b \in A_1$. Functors X and A are defined essentially as in Priestley duality. That is, for $(\mathbf{A}, \nabla, \Delta) \in \mathsf{Twist}$, we let $X(\mathbf{A}, \nabla, \Delta) := \langle X(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq, C_{\mathbf{A}}, O_{\mathbf{A}} \rangle$ and, for a Nelson-Esakia space $\langle X, \leq, \tau, C, O \rangle$, we let $A(\langle X, \leq, \tau, C, O \rangle) = \langle A(X), \nabla_C, \Delta_O \rangle$. On morphisms the two functors are defined set-theoretically exactly as in Priestley duality.

As mentioned above, our duality between twist-structures and Nelson-Esakia spaces can be extended to other classes of algebras.

One example is (non-bounded) Brouwerian lattices. In this case all that we need is to extend Esakia duality to Brouwerian lattices. This is relatively straightforward, because any Brouwerian lattice $\mathbf{A}$ can be extended to a Heyting algebra $\mathbf{A}^*$ in a canonical way by simply adding a new bottom element, regardless of whether **A** already had one or not. We then look at the dual Esakia space of $\mathbf{A}^*$, whose underlying poset will necessarily have a maximum element (corresponding to the set A, which is a prime filter of $\mathbf{A}^*$). Conversely, to any Esakia space with maximum we associate a Brouwerian lattice by considering the algebra of non-empty clopen up-sets. Everything else works as in Esakia duality, and the duality between Brouwerian lattices and Esakia spaces with maximum is extended to twist-structures over Brouwerian lattices and Nelson-Esakias spaces with maximum by taking care of the additional structure (∇, Δ, C, O) in the way described above.

The case of N4-lattices with monotone modal operators is more involved, but the strategy is essentially the same. These enriched algebras are also representable as twist-structures $(\mathbf{A}, \nabla, \Delta)$, where $\mathbf{A}$ is a Brouwerian lattice (or a Heyting or a Boolean algebra) which is itself endowed with modal operators, for instance suppose $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \Box, \Diamond \rangle$. Then the reduct $\langle A, \wedge, \vee, \Box, \Diamond \rangle$ is a distributive lattice with modal operators, a structure for which a duality theory is already available (see, e.g., [5, 6]). Thus, in order to extend this to a topological duality for "modal twist-structures" (and, therefore, for N4-lattices with modal operators), we only need to take care of the interaction between the modal operators (which are represented, on the topological side, by relations or by neighborhood functions on the dual space) and the additional structure given by the sets ∇, Δ, C, O. Besides its intrinsic interest, such an investigation sheds further light on the semantics of modal expansions of Belnap-Dunn logic; for instance, it has enabled us to introduce a state-based semantics for the non-normal paraconsistent modal logic of [14].

References

[1] A. Almukdad and D. Nelson. Constructible falsity and inexact predicates. *The Journal of Symbolic Logic*, 49(1):231–233, 1984.

[2] W. H. Cornish and P. R. Fowler. Coproducts of De Morgan algebras. *Bull. Austral. Math. Soc.*, 16(1):1–13, 1977.

[3] W. H. Cornish and P. R. Fowler. Coproducts of Kleene algebras. *J. Austral. Math. Soc. Ser. A*, 27(2):209–220, 1979.

[4] L. Esakia. Topological Kripke models. *Soviet Math. Dokl.*, 15:147–151, 1974.

[5] R. Goldblatt. Varieties of Complex Algebras. *Annals of Pure and Applied Logic*, 44(3):173–242, 1989.

[6] H. H. Hansen. Monotonic modal logics. Research Report PP-2003-24, Institute for Logic, Language and Computation, University of Amsterdam, Amsterdam, 2003.

[7] D. Nelson. Constructible falsity. *The Journal of Symbolic Logic*, 14:16–26, 1949.

[8] S. P. Odintsov. Algebraic semantics for paraconsistent Nelson's logic. *Journal of Logic and Computation*, 13(4):453–468, 2003.

[9] S. P. Odintsov. On the representation of N4-lattices. *Studia Logica*, 76(3):385–405, 2004.

[10] S. P. Odintsov. Priestley duality for paraconsistent nelson's logic. *Studia Logica*, 96(1):65–93, 2010.

[11] S. P. Odintsov and E. I. Latkin. BK-lattices. Algebraic semantics for Belnapian modal logics. *Studia Logica*, 100(1-2):319–338, 2012.

[12] S. P. Odintsov and H. Wansing. Modal logics with Belnapian truth values. *Journal of Applied Non-Classical Logics*, 20:279–301, 2010.

[13] H. Rasiowa. *An algebraic approach to non-classical logics*, volume 78 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1974.

[14] U. Rivieccio. Paraconsistent modal logics. *Electronic Notes in Theoretical Computer Science*, 278:173–186, 2011.

[15] M. Spinks and R. Veroff. Paraconsistent constructive logic with strong negation is a contraction-free relevant logic. *In preparation*.