Open projections do not form a right residuated lattice

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A lattice L with operations $\odot, \multimap$ is called *right residuated* if

$$x \odot y \leq z \;\Leftrightarrow\; x \leq y \multimap z$$

for all $x, y, z \in L$. (So we do not require the dual arrow nor the associativity of $\odot$.) In other words, the actions $- \odot y, y \multimap - : L \to L$, for any fixed $y \in L$, provide a Galois connection on L.

The Galois connection is also completely determined by the partial isomorphism between fixpoints of closure mapping $y \multimap (- \odot y)$ and coclosure mapping $(y \multimap -) \odot y$. Thus we can restrict $\odot, \multimap$ to partial operations $\cdot, \to$ where $x \cdot y$ is defined if $x = y \multimap (x \odot y)$ and $y \to z$ is defined if $z = (y \multimap z) \odot y$. The partial isomorphism provides an equivalence

$$x \cdot y = z \;\Leftrightarrow\; x = y \to z. \tag{$*$}$$

(We use a similar idea as in [6] but there the partial residuation law use standard inequalities. Cf. this also with [3].)

Conversely, whenever we have such a *partial right residuated lattice*, i.e. L satisfies $(*)$, for each y the $\cdot$-compatible elements define a closure $x \mapsto \hat{x}$ and the $\to$-compatible elements define a coclosure $z \mapsto \check{z}$, then by putting $x \odot y = \hat{x} \cdot y$ and $y \multimap z = y \to \check{z}$ we get a total right residuated structure on L.

1 Example. (1) Every orthomodular lattice with Sasaki operations

$$x \odot y = (x \vee y^{\perp}) \wedge y, \qquad\qquad y \multimap z = y^{\perp} \vee (y \wedge z)$$

is right residuated. The partial product $x \cdot y$ is defined for $x \geq y^{\perp}$, and $y \to z$ is defined for $z \leq y$. Thus $\cdot$ is a restriction of the meet operation on compatible elements.

(2) In the MV-chain $[0, 1]$ with $x \odot y = \max\{0, x + y - 1\}$ the partial product is defined also for $x \geq y^{\perp} = 1 - y$, i.e. whenever $x + y - 1 \geq 0$.

Note that in orthomodular lattices $\odot$ can be recovered from the meet defined on all pairs of compatible elements and such a partial operation also provides the closure and coclosure mappings. In that sense $\cdot$ is the minimal such generating partial operation.

The partial operations $\cdot, \to$ may have better algebraical properties than the total operations $\odot, \multimap$, e.g. in orthomodular lattices $\cdot$ is associative and commutative while $\odot$ is not. Sometimes it can be useful to study right residuated (or even residuated) structures by the partial operations [2].

In [4] and [1] the authors considered the lattice of closed right ideals (or equivalently so called *open projections*) as a spectrum of non-commutative C*-algebra. The embedding of a C*-algebra to an enveloping W*-algebra provides a representation of the lattice to an orthomodular

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lattice. In commutative case, the lattice is a just frame (the topology given by Gelfand–Naimark duality) and so a residuated lattice (as a complete Heyting algebra).

The lattice of open projections itself is not sufficient to recover the C*-algebra but it is sufficient when it is equipped with a partial meet operation on compatible elements [5]. It is a natural question whether such a partial operation extends to a total right residuated operation. Using the above ideas I will show an example of C*-algebra where there is no such extension and which disproves a conjecture in [5] that a product of open projections is open.

Recall from [4] that a projection $p \in A^{**}$ (here A^{**} is the enveloping W*-algebra of C*-algebra A) is called *open* if it is a *support* of some $a \in A$, i.e. the smallest projection such that $ap = a$.

2 Example. Let A be a C*-algebra which elements are norm-convergent sequences of 2×2 matrices a_n together with their limits, denoted by a_∞, i.e. $((a_n)_{n\in\mathbb{N}}, a_\infty) \in A$ iff $a_n \to a_\infty$. The enveloping W*-algebra A^{**} simply contains all sequences and the element a_∞ need not be a limit of a_n. The corresponding orthomodular lattice is a countable power of the orthomodular lattice of subspaces of $\mathbb{C}^2$ and the Sasaki operations are calculated componentwise.

Sequence $a_n = \left(\begin{smallmatrix}1 & 0\\ 0 & 1/n\end{smallmatrix}\right), a_\infty = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ belongs to A and since all matrices a_n are regular, its support is $p_n = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), p_\infty = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$. Constant sequence $q_n = q_\infty = \frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ is already a projection in A.

Let us consider the projections $p = (p_n), q = (q_n)$ as elements of A^{**}. Complementary projection $\neg p$ is given by $\neg p_n = \left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right), \neg p_\infty = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$ and clearly is not open. But we also have $(q \odot p)_n = ((\neg p \vee q) \wedge p)_n = (\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right) \vee \frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)) \wedge \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) = \frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ while $(q \odot p)_\infty = (\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right) \vee \frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)) \wedge \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, thus $(q \odot p)$ is not open too.

Finally, let r^m, s^m be collections of sequences for each $m \in \mathbb{N}$ given by $r^m_n = s^m_n = \frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ for $n < m$, $r^m_n = s^m_n = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ for $n \geq m$, and $r^m_\infty = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ while $s^m_\infty = \left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)$.

Each projection r^m or s^m is open. But the componentwise intersection of r^m is not open because the limit component is $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ while all other components are $\frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$. Thus $\bigwedge r^m$ (calculated in A, i.e. the interior of the discussed intersection) is given by $(\bigwedge r^m)_n = \frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right), (\bigwedge r^m)_\infty = \left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)$, hence $\bigwedge r^m = \bigwedge s^m$.

Since $r^m, s^m \leq p$ for each m, "compatible arrows" $p \to r^m, p \to s^m$ are defined and $(p \to r^m)_n = (p \to s^m)_n = \frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ for $n < m$, $(p \to r^m)_n = (p \to s^m)_n = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ for $n \geq m$, and $(p \to r^m)_\infty = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ while $(p \to s^m)_\infty = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$. Notice that all $p \to r^m, p \to s^m$ are open and $(\bigwedge p \to r^m)_n = (\bigwedge p \to s^m)_n = \frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right), (\bigwedge p \to r^m)_\infty = \frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ while $(\bigwedge p \to s^m)_\infty = \left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)$. Thus $\bigwedge p \to r^m \neq \bigwedge p \to s^m$ and hence $p \multimap \bigwedge r^m$ can not exist because $p \multimap -$ should preserve all infima.

3 Corollary. *Product of open projections need not be open.*

4 Corollary. *The partial monoid structure on compatible open projections need not extend to a right residuated structure on all open projections.*

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