On Modal Components of the **S4**-logics

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Abstract

We consider the representation of each extension of the modal logic S4 as sum of two components. The first component in such a representation is always included in Grzegorczyk logic and hence contains "modal resources" of the logic in question, while the second one uses essentially the resources of a corresponding intermediate logic. We prove some results towards the conjecture that every S4-logic has a representation with the least component of the first kind.

1 Preliminaries

We consider the intuitionistic propositional logic **Int** and modal propositional logic **S4**, both defined with the postulated rule of substitution, along with the lattices of their normal consistent extensions, *NExt***Int** (*intermediate logics*) and *NExt***S4** (**S4**-*logics*), respectively. The lattice operations are the set intersection $\cap$ as *meet* and the deduction closure $\oplus$ as *joint*. Other logics from *NExt***Int** and *NExt***S4** will also appear in the sequel.

The mappings $\rho : NExt\mathbf{S4} \longrightarrow NExt\mathbf{Int}$ and $\tau : NExt\mathbf{Int} \longrightarrow NExt\mathbf{S4}$ were defined in [3]. It is well known that the former mapping is a lattice epimorphism and the latter is an embedding; see [3]. Another mapping, $\sigma : NExt\mathbf{Int} \longrightarrow NExt\mathbf{Grz}$, defined by the equality $\sigma L = \mathbf{Grz} \oplus \tau L$ for any $L \in NExt\mathbf{Int}$, where **Grz** is Grzegorczyk logic, is an isomorphism; see [1] and [2]. This, along with inequalities obtained in [3], implies that

$$\tau\rho M \subseteq M \subseteq \mathbf{Grz} \oplus \tau\rho M, \qquad \text{(\textit{Blok-Esakia inequality})}$$

for any logic $M \in NExt\mathbf{S4}$.

Thus it can be suggested that any $M \in NExt\mathbf{S4}$ is equal to $M^* \oplus \tau\rho M$ for some logic $M^* \subseteq \mathbf{Grz}$. Indeed, for M^* one can always take $M \cap \mathbf{Grz}$; see [4]. Furthermore, we have the following.

Let $\mathcal{L} = \{\tau L \mid L \in NExt\mathbf{Int}\}$. An unspecified element of $\mathcal{L}$ will be denoted by τ. Then any $M \in NExt\mathbf{S4}$ can be represented as $M = M^* \oplus \tau$, where $M^* \subseteq \mathbf{Grz}$, i.e. $\rho M^* = \mathbf{Int}$. In this representation of M, the first term, M^*, is called the *modal component* of M and the second term, τ, is its *assertoric* (or *superintuitionistic*) *component* (or τ-*component*). Such a representation of M we call a τ-*representation.*

It has been noticed [4] that the assertoric component of M is uniquely determined by M and equals $\tau\rho M$, but its modal component may vary. Given an **S4**-logic M, the modal components of M constitute a dense sublattice of *NExt***S4** with the top element $M \cap \mathbf{Grz}$. This on-going research aims at proving the conjecture: Every **S4**-logic has a least modal component.

2 Examples of the modal components of some S4-logics

Below one can see different situations related to modal components of some **S4**-logics.

- Each logic in [**S4**, **Grz**] itself is its only modal component.
- All logics $\tau \in \mathcal{L}$ have **S4** their only modal component.
- If $\mathbf{Grz} \subseteq \mathbf{S4.1} \oplus \tau$ then the logics of [**S4.1**, **Grz**] constitute all the modal components of $\mathbf{Grz} \oplus \tau$.

In the sequel we obtain more examples.

3 *S*-series slicing of *NExt*S4

We arrange the Scroggs logics [5] as follows:

$$\mathbf{S5} = S_0 \subset \ldots \subset S_2 \subset S_1 = \mathbf{S4} + p \to \Box p. \qquad (S\text{-}series)$$

Definition 3.1 (S-series slicing)**.** *A logic M belongs to the nth S-slice, $n \geq 1$, if $M \subseteq S_n$ and $M \not\subseteq S_{n+1}$. If $M \subseteq S_n$, for all $n \geq 1$, that is to say, $M \in [\mathbf{S4}, S_0]$, then M lies in the 0th S-slice. We denote the nth S-slice by $\mathscr{S}_n$, $n \geq 0$.*

Thus $\mathscr{S}_0 = [\mathbf{S4}, \mathbf{S5}]$. Also, it is obvious that $\{\mathscr{S}_n\}_{n \geqslant 0}$ is a partition of *NExt***S4**. As well known, S_n, $n \geqslant 1$, is the logic of an n-atomic finite interior algebra with only two open elements. We denote such an algebra by B_n.

Definition 3.2 (logics K_n)**.** *Let χ_n be the characteristic formula of algebra B_n, $n \geq 1$. We define $K_n = \mathbf{S4} + \Box\chi_{n+1}$, for $n > 0$, and $K_0 = \mathbf{S4}$.*

Proposition 3.1. *Each S-slice is an interval. For $n \geq 1$, logic S_n is the top of the* n*th slice and logic K_n is its bottom. In particular, $\mathscr{S}_1 = [\mathbf{S4.1}, S_1]$.*

Corollary 3.1.1. *For each $n \geq 1$, (K_n, S_{n+1}) is a splitting pair in NExt***S4***.*

Proposition 3.2. *Let $\tau \in \mathcal{L}$. All logics from the nth S-slice having τ as their τ-component constitute the interval $[K_n \oplus \tau, M_n \oplus \tau]$.*

In addition, we prove the following:

- $K_{n+1} \subset K_n$ for any $n \geq 1$; and
- $\bigcap_{n \geqslant 1} K_n = \mathbf{S4}$.

4 *M*-series slicing of [S4, Grz]

We will be using the following notation:

$$M_0 = \mathbf{Grz} \cap \mathbf{S5} \text{ and } M_1 = \mathbf{Grz}.$$

We note [4] that the interval $[M_0, M_1]$ is ordered by $\subset$ in type $1 + \omega^*$:

$$M_0 \subset \ldots \subset M_2 \subset M_1, \qquad (M\text{-}series)$$

where $M_n = M_1 \cap S_n$ and $\bigcap_{n \geqslant 1} M_n = M_0$. To this, we add the following:

- $M_n \cap S_l$, whenever $M_l \subseteq M_n$ or $S_l \subseteq S_n$;
- $K_n \oplus S_l = S_n$, whenever $S_l \subseteq S_n$;

- $K_n \oplus M_l = M_n$, whenever $M_l \subseteq M_n$;
- $M_n \cap S_{n+1} = M_{n+1}$, for any $n \geqslant 1$.

Definition 4.1 (M-series slicing)**.** *A logic M from $[\mathbf{S4}, \mathbf{Grz}]$ belongs to the nth M-slice if and only if M is in the nth S-slice. In other words, the nth M-slice equals $[K_n, S_n] \cap [\mathbf{S4}, \mathbf{Grz}]$. We denote the nth M-slice, $n \geq 0$, by $\mathscr{E}_n$.*

We prove that for any $M \in [\mathbf{S4}, \mathbf{Grz}]$ and $n \geq 0$, the following conditions are equivalent:

$$\begin{array}{ll} a) & M \in \mathscr{E}_n; \\ b) & M \in [K_n, M_n]; \\ c) & M \oplus M_0 = M_n. \end{array}$$

For any $n \geqslant 1$, each of $(a) - (c)$ is equivalent to:

$$d) \quad M \subseteq M_n \text{ and } M \nsubseteq M_{n+1}.$$

Proposition 4.1. *Let us fix $n \geqslant 0$. If a modal logic M lies in $\mathscr{S}_n$, then any its modal component M^* belongs to $\mathscr{E}_n$. Conversely, for any modal logic M^* in $\mathscr{E}_n$ and any τ in $\mathcal{L}$, the logic $M^* \oplus \tau$ lies in $\mathscr{S}_n$.*

From Proposition 4.1 and some properties mentioned above we derive:

- For any $n \geqslant 0$, all modal logics of $\mathscr{E}_n$ are the modal components of the logic S_n.

Also, we obtain the following: Given $\tau \in \mathcal{L}$,

- if $M_n \subseteq K_n \oplus \tau$ then the logics of $[K_n, M_n]$ constitute all the modal components of $M_n \oplus \tau$;

5 Least modal components

In this section we will show that the existence of the least modal component of a logic M can be reduced to the question of definability of some function of M. The proposition of this section states that the definability of this function should be checked for some logics of the 0th S-slice.

Definition 5.1 (Mappings h_n, h_{n0}, and g_{n0})**.** *For any $n \geqslant 0$, we define: $h_n : M \mapsto M \cap S_n$, where $M \in NExt\mathbf{S4}$. We denote by h_{n0} and by g_{n0} the mapping h_0 restricted to $\mathscr{S}_n$ and $\mathscr{E}_n$, respectively.*

We observe the following:

- h_{n0} a lattice embedding of $\mathscr{S}_n$ into $\mathscr{S}_0$;
- g_{n0} is a lattice embedding of $\mathscr{E}_n$ into $\mathscr{E}_0$.

Proposition 5.1. *Given an $\mathbf{S4}$-logic $M \in \mathscr{S}_n$, M has a least modal component if and only if $h_{n0}(M)$ has it.*

Definition 5.2 (difference operation $d(X, Y)$)**.** *Given two logics X and Y, a logic $C \subseteq X$ is called the difference of the subtraction of Y from X, if for any logic Z, the following equivalence holds:*

$$C \subseteq Z \subseteq X \Leftrightarrow X = Z \oplus Y.$$

If such C exists for given X and Y, it is obviously unique. We denote it by $d(X, Y)$.

The operation $d(X,Y)$ is certainly partial. For instance, $d(M_0, \mathbf{Dum})$ is undefined, where $\mathbf{Dum} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to (\Diamond\Box p \to p)$.

Next we define: Given $M \in NExt\mathbf{S4}$,

$$d^*(M) = d(M, \tau\rho M).$$

Proposition 5.2. *Given $M \in NExt\mathbf{S4}$, $d^*(M)$ is defined if and only if $d^*(M)$ is the least modal component of M.*

The next theorem shows that our search for the definability of the d^* function on $NExt\mathbf{S4}$ can be reduces to the 0th S-slice.

Proposition 5.3. *Every* **S4***-logic has its least modal component if and only if for any $M^* \in [\mathbf{S4}, M_0]$ and $\tau \in \mathcal{L}$, $d(M^*, \tau \cap M^*)$ is defined, or, equivalently, $d^*(M^* \oplus \tau)$ is defined, providing that $\tau \cap \mathbf{Grz} \subseteq M^*$.*

6 Greatest modal components

We remind the reader that any logic $M \in NExt(\mathbf{S4})$ has its greatest modal component which is $M \cap \mathbf{Grz}$. Also, we know from Proposition 3.2 that all **S4**-logics of the nth S-slice that have a logic $\tau \in \mathcal{L}$ constitute the interval $[K_n \oplus \tau, M_n \oplus \tau]$. The next proposition reads that the greatest modal components of the logics of the last set form an interval.

Proposition 6.1. *Let $\tau \in \mathcal{L}$. The greatest modal components of all logics from the nth S-slice having τ as their τ-component constitute the interval $[K_n \oplus (\mathbf{Grz} \cap \tau), M_n]$.*

References

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