Continued Fraction Representation of the Generalized Operator Entropy

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Abstract

The direct calculation of the Generalized operator entropy proves difficult by the appearance of rational exponents of matrices. The main motivation of this work is to overcome these difficulties and to present a practical and efficient method for this calculation using its representation by the matrix continued fraction. At the end of our paper, we deduce a continued fraction expansion of the Bregman operator divergence.

Keywords: Continued fraction, positive definite matrix, generalized operator entropy, divergence operator.

1 Introduction and motivation

The basic idea of the continued fraction theory over real numbers is to give an approximation of various real numbers by the rational. One of the main reasons why continued fractions are so useful in computation is that they often provide representation for transcendental functions that are much more generally valid than the classical representation by, say, the power series. Further; in the convergent case, the continued fractions expansions have the advantage that they converge more rapidly than other numerical algorithms.

Recently, the extension of continued fractions theory from real numbers to the matrix case has seen several development and interesting applications, [6]. Since calculations involving matrix valued functions with matrix arguments are feasible with large computers, it will be an interesting attempt to develop such matrix theory. The real case is relatively well studied in the literature [8, 9]. However, in contrast to the theoretical importance, one can find in mathematical literature only a few results on the continued fractions with matrices arguments [11, 14].

The theory of operator means for positive and bounded linear operators on a Hilbert space was initiated by T. Ando [1] and established by him and F. Kubo in connection with Loweners theory for the operator monotone functions. It is started from the presence of the notion of parallel sum as a tool for analyzing multi-port electrical networks in engineering, see [2, 3].

In 1850, Clausius, introduced the notion of entropy in thermodynamics. Since then several extensions and reformulations have been developed in various disciplines [12, 15]. There have been investigated the so called entropy inequalities by some mathematicians, see [16, 17] and references therein.

The relative operator entropy of strictly positive operators A, B was introduced in noncommutative information theory by Fujii and Kamei [4], it is defined by

$$S(A \mid B) = A^{1/2} \ln(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

For any real number $q \in \mathbb{R}$, We also consider a path

$$f_q(A, B) = A^{1/2} (A^{-1/2} B A^{-1/2})^q A^{1/2}.$$

If $0 < q < 1$, the previous path $f_q(A, B)$ is the generalized geometric mean of A and B.

In the present paper, we also study the representation of the generalized operator entropy which is defined for two positive operators A and B on a Hilbert space and any real number $q \in]0, 1[$, by

$$S_q(A \mid B) = A^{1/2} (A^{-1/2} B A^{-1/2})^q \ln(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

The Bregman operator divergence introduced by Petz [15] is defined by

$$D(A \mid B) = B - A - S(A \mid B).$$

In [7], Isa et al. have generalized $D(A \mid B)$ as follows

$$D_q(A \mid B) = f_{-q}(B, A) - f_q(A, B) - S_q(A \mid B).$$

At the end of our paper, we also express the continued fraction representation of the divergence operator $D_q(A \mid B)$.
For simplicity and clearness, we restrict ourselves to positive definite matrices, but our results can be, without special difficulties, projected to the case of positive definite operators from an infinite dimensional Hilbert space into itself.

The computation of $S(A \mid B,)$ $S_q(A \mid B)$ and $D_q(A \mid B)$ from the original definitions impose many difficulties by virtue of the appearance of the rational exponents of the matrices. One fundamental of this paper is to remove this difficulty and reveal a practical method, involving matrix continued fraction.

2 Definitions and notations

The functions of matrix arguments play a widespreased role in science and engineering, with applications areas from nuclear magnetic resonance [1]. So for scalar polynomial $p(z) = \sum_{i=0}^{k} a_i z^i$ gives rise to a matrix polynomial with scalar coefficients by simply substitution A^i for z^i:

$$P(A) = \sum_{i=0}^{k} a_i A^i$$

More generally, for a function f with a series representation on an open disk containing the eigenvalues of A, we are able to define the matrix function $f(A)$ via the Taylor series for f, see [6].
Alternatively, given a function f that is analytic inside a closed contour Γ which enclose the eigenvalues of A, $f(A)$ can be defined, by analogy with Cauchy's integral theorem by

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1} dz.$$

The definition is known as the matrix version of Cauchy's integral theorem. Let $\mathcal{M}_m$ be the algebra of real square matrices, we now mention an important result of matrix functions.

Let $A \in \mathcal{M}_m$, A is said to be positive semi-definite (resp. positive definite) if A is symmetric and

$$\forall x \in \mathbb{R}^m, < Ax, x > \geq 0 \ \ (\text{resp. } \forall x \in \mathbb{R}^m x \neq 0, < Ax, x > > 0).$$

where $< ., . >$ denotes the standard scalar product of $\mathbb{R}^m$.

We observe that positive semi-definiteness induces a partial ordering on the space of symmetric matrices. Henceforth, whenever we say that $A \in \mathcal{M}_m$ is positive semi-definite (or positive definite), it will be assumed that A is symmetric.

For any $A, B \in \mathcal{M}_m$ with B invertible, we write $\dfrac{A}{B} = B^{-1}A$, in particular, if $A = I$, where I is the m^{th} order identity matrix, then $\dfrac{I}{B} = B^{-1}$. If $A \neq 0$, it is clear that for any invertible matrix C which don't commute with A and B, we have

$$\frac{CA}{CB} = \frac{A}{B} \neq \frac{AC}{BC}.$$

Definition 1 Let $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ be two nonzero sequences of $\mathcal{M}_m$. The continued fraction of (A_n) and (B_n) denoted by $K(B_n/A_n)$ is the quantity

$$A_0 + \cfrac{B_1}{A_1 + \cfrac{B_2}{A_2 + \cdots}} = \left[A_0;\ \frac{B_1}{A_1}, \frac{B_2}{A_2}, \cdots\right].$$

Sometimes, we use briefly the notation $\left[A_0;\ \frac{B_n}{A_n}\right]_{n=1}^{+\infty}$. The fractions $\frac{B_n}{A_n}$ and $\frac{P_n}{Q_n} = \left[A_0;\ \frac{B_k}{A_k}\right]_{k=1}^{n}$ are called, respectively, the n^{th} partial quotient and the n^{th} convergent of the continued fraction $K(B_n/A_n)$.
When $B_n = I$ for all $n \geq 1$, then $K(I/A_n)$ is called a simple continued fraction.

We now introduce some topological notion of continued fractions with matrix arguments. Let $A \in \mathcal{M}_m$, we put

$$\|A\| = \sup_{x\neq 0}\{\frac{\|Ax\|}{\|x\|}\} = \sup_{\|x\|=1}\{\|Ax\|\}.$$

Let (A_n) be a sequence of matrices in $\mathcal{M}_m$.We say that (A_n) converges in $\mathcal{M}_m$ if there exists a matrix $A \in \mathcal{M}_m$ such that $\lim_{n\to+\infty} \|A_n - A\| = 0$.

The continued fraction $K(B_n/A_n)$ converges in $\mathcal{M}_m$ if the sequence $(F_n) = (\frac{P_n}{Q_n})$ converges in $\mathcal{M}_m$ in the sense that there exists a matrix $F \in \mathcal{M}_m$ such that $\lim_{n\to+\infty} F_n - F = 0$. In this case, we note

$$F = \left[A_0;\ \frac{B_n}{A_n}\right]_{n=1}^{+\infty}.$$

We note that the evaluation of n^{th} convergent according to the Definition 2.1 is not practical because we have to repeat inverse matrix. The following proposition gives an adequate method to calculate $K(B_n/A_n)$.

Proposition 1 *The elements $(P_n)_{n\geq -1}$ and $(Q_n)_{n\geq -1}$ of the n^{th} convergent of $K(B_n/A_n)$ are given by the relationships*

$$\begin{cases} P_{-1} = I, & P_0 = A_0 \\ Q_{-1} = 0, & Q_0 = I \end{cases} \quad \textit{and} \quad \begin{cases} P_n = A_n P_{n-1} + B_n P_{n-2} \\ Q_n = A_n Q_{n-1} + B_n Q_{n-2} \end{cases}, \qquad n \geq 1.$$

Proof We prove it by induction. □

The proof of the next Proposition is elementary and we left it to the reader.

Proposition 2 *(i) For any two matrices C and D with C invertible, we have*

$$C\left[A_0;\ \frac{B_k}{A_k}\right]_{k=1}^{n} D = \left[CA_0D\ ;\ \frac{B_1D}{A_1C^{-1}};\ \frac{B_2C^{-1}}{A_2},\frac{B_k}{A_k}\right]_{k=3}^{n}.$$

(ii) If two matrices A and B are similar, with $A = XBX^{-1}$, then $f(A)$ and $f(B)$ are similar and we have $f(A) = Xf(B)X^{-1}$.

Definition 2 Let $(A_n), (B_n), (C_n)$ and (D_n) be four matrix sequences in $\mathcal{M}_m$. We say that the continued fractions $K(B_n/A_n)$ and $K(D_n/C_n)$ are equivalent if we have $F_n = G_n$ for all $n \geq 1$, where F_n and G_n are the n^{th} convergent of $K(B_n/A_n)$ and $K(D_n/C_n)$ respectively.

In order to simplify the statement on some partial quotients of continued fractions with matrix arguments, we need the following proposition which is an example of equivalent continued fractions.

Proposition 3 *Let $\left[A_0;\ \frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ be a given continued fraction. Then we have*

$$\frac{P_n}{Q_n} = \left[A_0;\ \frac{B_k}{A_k}\right]_{k=1}^{n} = \left[A_0;\ \frac{X_kB_kX_{k-2}^{-1}}{X_kA_kX_{k-1}^{-1}}\right]_{k=1}^{n}$$

where $X_{-1} = X_0 = I$ and $X_1,\ X_2,\ \cdots,X_n$ are arbitrary invertible matrices.

Proof Let $\frac{P_n}{Q_n}$ and $\frac{\widetilde{P}_n}{\widetilde{Q}_n}$ be the n^{th} convergent of the continued fractions $\left[A_0;\ \frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ and $\left[A_0;\ \frac{X_kB_kX_{k-2}^{-1}}{X_kA_kX_{k-1}^{-1}}\right]_{k=1}^{+\infty}$ respectively. By proposition 2, for all $n \geq 1$, we can write

$$\widetilde{P}_n = X_nA_nX_{n-1}^{-1}\widetilde{P}_{n-1} + X_nB_nX_{n-2}^{-1}\widetilde{P}_{n-2},$$

which is equivalent to

$$X_n^{-1}\widetilde{P}_n = A_n\left(A_nX_{n-1}^{-1}\widetilde{P}_{n-1}\right) + B_n\left(X_{n-2}^{-1}\widetilde{P}_{n-2}\right).$$

This last result joined to the initial conditions prove that for all $n \geq 1$, $X_{n-1}^{-1}\widetilde{P}_n = P_n$. A similar result can be obtained for Q_n. Consequently, both continued fractions have the same convergent and Proposition 3 follows. □

Definition 3 Let A be a positive definite matrix in $\mathcal{M}_m, X \in \mathcal{M}_m$ and α a real number such that $0 < \alpha < 1$. We define the matrix A^α by the formulae

$$A^\alpha = \exp(\alpha \ln A),$$

where " exp " is the matrix exponential given by the series

$$\exp(X) = \sum_{n=0}^{+\infty} \frac{X^n}{n!}$$

and " ln " is the neperian logarithm defined by

$$\ln A = -2 \sum_{n=0}^{+\infty} \frac{1}{2n+1} \left(\frac{I-A}{I+A} \right)^{2n+1}.$$

3 Main results

3.1 Continued fraction representation of $A^q \ln(A)$,

This section is devoted to give a continued fraction representation of $A^q \ln(A)$, where A is a positive definite matrix.

Theorem 4 *Let $A \in \mathcal{M}_m$ be a positive definite matrix and q a positive real number such that $0 < q < 1$. If we put $A^q = \left[I; \frac{I}{A_k} \right]_{k=1}^{+\infty}$, $\ln A = \left[0; \frac{I}{\widetilde{A}_k} \right]_{k=1}^{+\infty}$ and $\varphi(A) = \frac{I-A}{I+A}$ then, the continued fraction expansions of $A^q \ln(A)$ is given by*

$$A^q \ln(A) = \left[0; \frac{A_1 + I}{A_1 \widetilde{A}_1}, \frac{A_1 \widetilde{A}_1 F_2}{E_2 - F_2}, \frac{E_{n-1} F_n}{E_n - F_n} \right]_{n=3}^{+\infty}$$

where

$$\begin{cases} A_1 = \frac{1}{2q}(-I - q\varphi(A))\varphi(A)^{-1}, \ A_2 = \frac{-6q}{(q^2-1)}\varphi(A)^{-1}, \\ A_{2k} = \frac{-2q(q^2-2^2)\cdots(q^2-(2k-2)^2)}{(q^2-1)\cdots(q^2-(2k-1)^2)}(4k-1)\varphi(A)^{-1}, \quad k \geq 2, \\ A_{2k+1} = \frac{-(q^2-1)\cdots(q^2-(2k-1)^2)}{2q(q^2-2^2)\cdots(q^2-4k^2)}(4k+1)\varphi(A)^{-1}(A), k \geq 1. \end{cases} \tag{1}$$

For all $n \geq 1$, the expression of $\widetilde{A}_n$ are given by the next relationships.

$$\begin{cases} \widetilde{A}_1 = (-2\varphi(A))^{-1}, \widetilde{A}_2 = \dfrac{6I}{\varphi(A)} \\ \widetilde{A}_{2k} = \dfrac{2^2.4^2 \cdots (2k-2)^2}{3^2.5^2 \cdots (2k-1)^2} \, 2(4k-1)\varphi(A)^{-1}, \quad k \geq 2 \\ \widetilde{A}_{2k+1} = \dfrac{3^2.5^2 \cdots (2k-1)^2}{2^2.4^2 \cdots (2k)^2} \, (-(4k+1))\varphi(A)^{-1}, \quad k \geq 1. \end{cases} \tag{2}$$

.

We also define

$$\begin{cases} E_n = Q_n\widetilde{Q}_n(Q_{n-2}P_{n-1} + \widetilde{Q}_{n-1}\widetilde{P}_{n-2}), \\ F_n = Q_{n-2}\widetilde{Q}_{n-2}(Q_nP_{n-1} + \widetilde{Q}_{n-1}\widetilde{P}_n). \end{cases}$$

The matrices P_n and Q_n (resp.$\widetilde{P}_n$) and $\widetilde{Q}_n$) are the numerator and denominator of the n^{th} convergent of A^q (resp. $\ln(A)$) which are defined for all $n \geq 1$ by

$$\begin{cases} P_n = A_nP_{n-1} + P_{n-2}, \\ Q_n = A_nQ_n + Q_{n-2}, \end{cases} \quad \textit{and} \quad \begin{cases} \widetilde{P}_n = \widetilde{A}_n\widetilde{P}_{n-1} + \widetilde{P}_{n-2}, \\ \widetilde{Q}_n = \widetilde{A}_n\widetilde{Q}_{n-1} + \widetilde{Q}_{n-2}. \end{cases}$$

In order to prove Theorem 4, we begin by studying the real case.

3.1.1 Real case

We begin by giving some lemmas concerning the real continued fraction which are important in the sequel. The following lemma characterizes equivalence of continued fractions.

Lemma 5 ([8]) *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 1}$ be two non-zero sequences of real numbers. The continued fractions*

$$\left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, .., .\frac{b_n}{a_n}, \cdots\right] \quad \textit{and} \quad \left[a_0; \frac{1}{a_1^*}, \frac{1}{a_2^*}, \cdots, \frac{1}{a_n^*}, \cdots\right]$$

are equivalent, where

$$\begin{cases} a_1^* = \dfrac{a_1}{b_1},\ a_2^* = \dfrac{b_1}{b_2}a_2, \\ a_{2k}^* = \dfrac{b_1b_3\cdots b_{2k-1}}{b_2b_4\cdots b_{2k}}a_{2k}, \quad k\geq 2, \\ a_{2k+1}^* = \dfrac{b_2b_4\cdots b_{2k}}{b_1b_3\cdots b_{2k+1}}a_{2k+1}, \quad k\geq 1. \end{cases}$$

.

We now give a lemma which expresses the n^{th} convergent for the product of two continued fractions.

Lemma 6 ([11]) *Let C and D be two real continued fractions which are defined by*

$$C = \left[c_0; \frac{1}{c_1}, \frac{1}{c_n}\right]_{n=2}^{+\infty}, \ D = \left[d_0; \frac{1}{d_1}, \frac{1}{d_n}\right]_{n=2}^{+\infty},$$

where c_k *and* d_k *are non-zero real numbers. If we put*

$$C_n = \left[c_0;\ \frac{1}{c_1}, \frac{1}{c_2}, \cdots, \frac{1}{c_n}\right] = \frac{{}^{c}p_n}{{}^{c}q_n} \quad \textit{and} \quad D_n = \left[d_0;\ \frac{1}{d_1}, \frac{1}{d_2}, \cdots, \frac{1}{d_n}\right] = \frac{{}^{d}p_n}{{}^{d}q_n},$$

then, for all $n \geq 1$, *we have*

$$C_n D_n = \left[c_0 d_0;\ \frac{c_0 c_1 + d_0 d_1 + 1}{c_1 d_1}, \frac{c_1 d_1 f_2}{e_2 - f_2}, \frac{e_2 f_3}{e_3 - f_3}, \cdots, \frac{e_{n-1} f_n}{e_n - f_n}\right],$$

where

$$\begin{cases} e_n = {}^{c}q_n\ {}^{d}q_n\left({}^{c}q_{n-2}\ {}^{c}p_{n-1} + {}^{d}q_{n-1}\ {}^{d}p_{n-2}\right), \\ f_n = {}^{c}q_{n-2}\ {}^{d}q_{n-2}\left({}^{c}q_n\ {}^{c}p_{n-1} + {}^{d}q_{n-1}\ {}^{d}p_n\right). \end{cases}$$

The following Lemma gives two equivalent continued fraction expansions of λ^q, where λ and q are two strictly positive real numbers.

Lemma 7 *(i) Let* λ *and* q *be two positive real numbers,* $\varphi(\lambda) = \frac{1-\lambda}{1+\lambda}$. *The continued fraction expansions of* λ^q *is*

$$\lambda^q = \left[1;\ \frac{2q\varphi(\lambda)}{-1 - q\varphi(\lambda)}, \frac{\left(q^2 - (k-1)^2\right)\varphi^2(\lambda)}{-(2k-1)}\right]_{k=2}^{+\infty}.$$

(ii) The simple continued fraction of λ^q *is given by*

$$\lambda^q = \left[1;\ \frac{1}{c_1^*}, \frac{1}{c_2^*}, \cdots, \frac{1}{c_n^*}, \cdots\right] \quad \textit{where}$$

$$\begin{cases} c_1^* = \dfrac{-1 - q\varphi(\lambda)}{2q\varphi(\lambda)},\ c_2^* = \dfrac{-6q}{(q^2-1)\varphi(\lambda)} \\ c_{2k}^* = \dfrac{-2q\left(q^2 - 2^2\right)\cdots\left(q^2 - (2k-2)^2\right)}{(q^2-1)\cdots\left(q^2 - (2k-1)^2\right)}(4k-1)\,\varphi(\lambda)^{-1}, \quad k \geq 2 \\ c_{2k+1}^* = \dfrac{-\left(q^2 - 1\right)\cdots\left(q^2 - (2k-1)^2\right)}{2q\left(q^2 - 2^2\right)\cdots\left(q^2 - (2k)^2\right)}(4k+1)\,\varphi^{-1}(\lambda), \quad k \geq 1. \end{cases}$$

Proof (i) See [13].
(ii) By appropriate iteration and by applying Lemma 5 we prove it. □

Lemma 8 *(i) Let* λ *be a real number such that* $\lambda > 0$, $\lambda \neq 1$ *and* $\varphi(\lambda) = \frac{1-\lambda}{1+\lambda}$. *A continued fraction expansion of* $\ln(\lambda)$ *is given by*

$$\ln(\lambda) = \left[0;\ \frac{-2\varphi(\lambda)}{1}, \frac{-\varphi(\lambda)^2}{3}, \frac{-2^2\varphi(\lambda)^2}{5}, \frac{-n^2\varphi(\lambda)^2}{2n+1}\right]_{n=3}^{+\infty}.$$

(ii) The simple continued fraction of $\ln(\lambda)$ *is*

$$\ln(\lambda) = \left[0;\ \frac{1}{d_1^*}, \frac{1}{d_2^*}, \cdots, \frac{1}{d_n^*}, \cdots\right] \quad \textit{where}$$

$$\begin{cases} d_1^* = \dfrac{1}{-2\varphi(\lambda)}, \quad d_2^* = \dfrac{6}{\varphi(\lambda)}, \\ d_{2k}^* = \dfrac{2^2.4^2.\cdots(2k-2)^2}{1^2.3^2.5^2.\cdots(2k-1)^2}(2(4k-1))\varphi(\lambda)^{-1}, \quad k \geq 2, \\ d_{2k+1}^* = \dfrac{1^2.3^2.5^2.\cdots(2k-1)^2}{2^2.4^2.\cdots(2k)^2}(-(4k+1))\varphi(\lambda)^{-1},\ k \geq 1. \end{cases}$$

.

Proof (i) See [14].
(ii) We deduce it by applying Lemma 5. □

The next Theorem is a real version of the previous Theorem 4.

Theorem 9 *With the same notations as bellow, let* λ *and* q *be two strictly positive real numbers such that* $0 < q < 1$. *A continued fraction representation of the real* $\lambda^q \ln(\lambda)$ *is given by:*

$$\lambda^q\ \ln(\lambda) = \left[0;\ \frac{c_1^*+1}{c_1^* d_1^*}, \frac{(c_1^* d_1^*)f_2}{e_2 - f_2}, \cdots, \frac{e_{n-1}f_n}{e_n - f_n}\right]_{n=3}^{+\infty},$$

where

$$\begin{cases} e_n = q_n \widetilde{q}_n \left(q_{n-2}p_{n-1} + \widetilde{q}_{n-1}\widetilde{p}_{n-2}\right) \\ f_n = q_{n-2}\widetilde{q}_{n-2}\left(q_n p_{n-1} + \widetilde{q}_{n-1}\widetilde{p}_n\right). \end{cases}$$

p_n *and* q_n *(resp.* $\widetilde{p}_n$*) and* $\widetilde{q}_n$*) are numerator and denominator of the* n^{th} *convergent of* λ^q *(resp.* $\ln(\lambda)$*). They are defined by*

$$\begin{cases} p_n = c_n^* p_{n-1} + p_{n-2} \\ q_n = c_n^* q_{n-1} + q_{n-2} \end{cases}, \quad \begin{cases} \widetilde{p}_n = d_n^* \widetilde{q}_{n-1} + \widetilde{q}_{n-2} \\ \widetilde{q}_n = d_n^* \widetilde{q}_{n-1} + \widetilde{q}_{n-2}. \end{cases}$$

Proof We apply Lemmas 6, 7 and 8 by putting $C = \lambda^q$ and $D = \ln(\lambda)$. □

3.1.2 Proof of theorem of Theorem 4

Let $A \in \mathcal{M}_m$ be a positive definite matrix. Then there exists an invertible matrix X such that $A = XDX^{-1}$ where $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdot, \lambda_m)$ and $\lambda_i > 0$, for $1 \leq i < m$.
As the functions $f(z) = z^q$ and $g(z) = \ln(z)$ are continuous in the open interval $\mathbb{R}_+^*$, then we get

$$A^q \ln(A) = XD^q \ln(D)X^{-1}.$$

Let us define the sequences (P_n) and (Q_n) (the numerator and denominator of the n^{th} convergent of $D^q \ln(D)$ by

$$\begin{cases} P_{-1} = I, P_0 = 0, P_1 = D_1 + I, P_2 = (E_2' - F_2')(D_1 + I), \\ Q_{-1} = 0, Q_0 = I, Q_1 = D_1\widetilde{D}_1, Q_2 = (E_2' - F_2')Q_1 + D_1\widetilde{D}_1F_2', \end{cases}$$

and for $n \geq 3$,

$$\begin{cases} P_n = (E_n' - F_n')\, P_{n-1} + E_{n-1}'F_n'P_{n-2} \\ Q_n = (E_n' - F_n')\, Q_{n-1} + E_{n-1}'F_n'Q_{n-2} \end{cases}$$

$$\begin{cases} E_n' = Q_n\widetilde{Q}_n \left(Q_{n-2}P_{n-1} + \widetilde{Q}_{n-1}\widetilde{P}_{n-2}\right) \\ F_n' = Q_{n-2}\widetilde{Q}_{n-2} \left(Q_nP_{n-1} + \widetilde{Q}_{n-1}\widetilde{P}_n\right). \end{cases}$$

The matrices P_n, Q_n (resp. $\widetilde{P}_n, \widetilde{Q}_n$) are the numerator and denominator of the n^{th} convergent of D^q (resp. $\ln(D)$) which are defined by

$$\begin{cases} P_n = D_nP_{n-1} + P_{n-2} \\ Q_n = D_nQ_{n-1} + Q_{n-2} \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{P}_n = \widetilde{D}_n\widetilde{P}_{n-1} + \widetilde{P}_{n-2} \\ \widetilde{Q}_n = \widetilde{d}_n\widetilde{Q}_{n-1} + \widetilde{Q}_{n-2}. \end{cases}$$

We recall that

$$\begin{cases} D_1 = \dfrac{-I - q\varphi(D)}{2q\varphi(D)}, D_2 = \dfrac{6qI}{(q^2-1)\varphi(D)} \\ D_{2k} = \dfrac{-2q\left(q^2-2^2\right)\cdots\left(q^2-(2k-2)^2\right)}{(q^2-1)\cdots(q^2-(2k-1)^2)}(4k-1)\,\varphi^{-1}(D)\ ,\ \ k \geq 1 \\ D_{2k+1} = \dfrac{-\left(q^2-1\right)\cdots\left(q^2-(2k-1)^2\right)}{2q\left(q^2-2^2\right)\cdots(q^2-4k^2)}(4k+1)\,\varphi^{-1}(D)\,,\ \ k \geq 1. \end{cases}$$

and

$$\begin{cases} \widetilde{D}_1 = (-2\varphi(D))^{-1}, \widetilde{D}_2 = \dfrac{6I}{\varphi(D)}, \\ \widetilde{D}_{2k} = \dfrac{2\left(2.4.\cdots(2k-2)\right)^2}{\left(1.3.5.\cdots(2k-1)\right)^2}(4k-1)\varphi^{-1}(D), \quad k \geq 2 \\ \widetilde{D}_{2k+1} = \dfrac{\left(-1.3.5.\cdots(2k-1)\right)^2}{\left(2.4.\cdots(2k)\right)^2 \varphi(D)}(4k+1)\varphi^{-1}(D), \quad k \geq 1. \end{cases}$$

We see that P_n, Q_n, E'_n and F'_n are diagonal matrices, so we put

$$\begin{cases} P_n = diag\left(p_n^1,\ p_n^2, \cdots, p_n^m\right) \\ Q_n = diag\left(q_n^1,\ q_n^2, \cdots, q_n^m\right) \end{cases}, \quad \begin{cases} E'_n = diag\left(e_n^1,\ e_n^2, \cdots, e_n^m\right), \\ F'_n = diag\left(f_n^1,\ f_n^2, \cdots, f_n^m\right). \end{cases}$$

We obtain for each $1 \leq i \leq m$,

$$\begin{cases} p_{-1}^i = 1, p_0^i = 0, p_1^i = c_{i1}^* + 1, \quad p_2^i = (e_2^i - f_2^i)(c_{i1}^* + 1), \\ q_{-1}^i = 0, q_0^i = 1, q_1^i = c_{i1}^* d_{i1}^*, \quad q_2^i = (e_2^i - f_2^i) q_1^i + c_{i1}^* d_{i1}^* f_2^i, \end{cases}$$

and for $n \geq 3$, we have

$$\begin{cases} p_n^i = \left(e_n^i - f_n^i\right) p_{n-1}^i + e_{n-1}^i f_n^i p_{n-2}^i, \\ q_n^i = \left(e_n^i - f_n^i\right) q_{n-1}^i + e_{n-1}^i f_n^i q_{n-2}^i, \end{cases}$$

$$\begin{cases} e_n^i = q_n^i \widetilde{q}_n^i \left(q_{n-2}^i p_{n-1}^i + \widetilde{q}_{n-1}^i \widetilde{p}_{n-2}^i\right) \\ f_n^i = q_{n-2}^i \widetilde{q}_{n-2}^i \left(q_n^i p_{n-1}^i + \widetilde{q}_{n-1}^i \widetilde{p}_n^i\right). \end{cases}$$

By Lemma 3.2, we deduce that $\dfrac{p_n^i}{q_n^i}$ converges to $\lambda_i^q \ln(\lambda_i)$ for $1 \leq i \leq m$. It follows that the matrix $\dfrac{P_n}{Q_n}$ converges to $D^q \ln(D)$. So, we get

$$D^q \ln(D) = \left[0; \frac{D_1 + I}{D_1 \widetilde{D}_1}, \frac{D_1 \widetilde{D}_1 F'_2}{E'_2 - F'_2}, \frac{E'_{n-1} F'_n}{E'_n - F'_n}\right]_{n=3}^{+\infty}.$$

By Proposition 3.2, we have

$$A^q \ln(A) = X(D^q \ln(D))X^{-1}$$

$$= \left[0;\ (\frac{D_1+I)X^{-1}}{D_1\widetilde{D}_1X^{-1}}, \frac{D_1\widetilde{D}_1F_2'X^{-1}}{E_2'-F_2'}, \frac{E_{n-1}'F_n'}{E_n'-F_n'} \right]_{n=3}^{+\infty}$$

Let us define the sequence $(X_n))_{n\geq -1}$ by $X_{-1}=X_0=I$, and for all $n\geq 1$, $X_n=X$. Then we have

$$\begin{cases} \frac{X_1(D_1+I)X^{-1}X_{-1}^{-1}}{X_1D_1\widetilde{D}_1X^{-1}X_0^{-1}} = \frac{A_1+I}{A_1\widetilde{A}_1}, \\ \frac{X_2(D_1\widetilde{D}_1F_2'X^{-1})X_0^{-1}}{X_2(E_2'-F_2')X_1^{-1}} = \frac{A_1\widetilde{A}_1F_2}{E_2-F_2},, \\ \frac{X_n(E_{n-1}'F_n')X_{n-2}^{-1}}{X_n(E_n'-F_n')X_{n-1}^{-1}} = \frac{E_{n-1}F_n}{E_n-F_n} \end{cases}$$

with $XE_n'X^{-1}=E_n$ and $XF_n'X^{-1}=F_n$ for all $n\geq 2$.

By applying the result of Proposition 3 to the sequence $(X_n)_{n\geq -1}$, we finish the proof of Theorem 4.

3.2 Representation of the generalized operator entropy

Theorem 10 *Let A and B be two positive definite matrices in $\mathcal{M}_m$, q a positive real number such that $0<q<1$. A continued fraction representation of the generalized operator entropy $S_q(A\mid B)$ is given by*

$$S_q(A\mid B) = \left[0;\ \frac{(A_1'+I)A^{1/2}}{A_1'\widetilde{A'}_1A^{-1/2}}, \frac{A_1'\widetilde{A'}_1F_2A^{-1/2}}{E_2-F_2}, \frac{E_{n-1}F_n}{E_n-F_n} \right]_{n=3}^{+\infty}$$

where $A_k'=A_k$ and $\widetilde{A'}_k=\widetilde{A}_k$ which are defined in Theorem 3.1 by the equalities (0.1) and (0.2) by replacing $\varphi(A)$ by

$$\varphi(A,B)=\varphi(A^{-1/2}BA^{-1/2}) = A^{1/2}(\frac{A-B}{A+B})A^{-1/2}.$$

Proof We have

$$(A^{-1/2}BA^{-1/2})^q\ln(A^{-1/2}BA^{-1/2}) = A^{-1/2}S_q(A\mid B)A^{-1/2}.$$

By applying Theorem 4 the continued fraction representation of $S_q(A\mid B)$ is

$$S_q(A\mid B) = A^{1/2}\left[0;\ \frac{(A_1'+I)}{A_1'\widetilde{A'}_1}, \frac{A_1'\widetilde{A'}_1F_2}{E_2-F_2}, \frac{E_{n-1}F_n}{E_n-F_n} \right]_{n=3}^{+\infty} A^{1/2}$$

That is

$$S_q(A\mid B) = \left[0;\ \frac{(A_1'+I)A^{1/2}}{A_1'\widetilde{A'}_1A^{-1/2}}, \frac{A_1'\widetilde{A'}_1F_2A^{-1/2}}{E_2-F_2}, \frac{E_{n-1}F_n}{E_n-F_n} \right]_{n=3}^{+\infty}$$

which completes the proof of Theorem 10. □

Let $q \in \mathbb{R}$, such that $0 < q < 1$ and n be an integer, we gave some properties of $S_q(A \mid B)$, which are shown in [7].

Lemma 11 *Let A and B be two positive definite matrices in $\mathcal{M}_m$, $n \in \mathbb{N}^*$, then we have*

$$(i) S_n(A \mid B) = (BA^{-1})^n S(A \mid B) = S(A \mid B)(A^{-1}B)^n.$$
$$(ii) S_{2n}(A \mid B) = (BA^{-1})^n S(A \mid B)(A^{-1}B)^n.$$
$$(iii) S_{2n+1}(A \mid B) = (BA^{-1})^n S_1(A \mid B)(A^{-1}B)^n.$$

Now we have the next result which gives a continued fraction expansions of the relative operator entropy $S_n(A \mid B)$.

Corollary 12 *Let A and B be two positive definite matrices in $\mathcal{M}_m$, $n \in \mathbb{N}$. A continued fraction representation of the generalized operator entropy $S_n(A \mid B)$ is given by*

$$S_n(A \mid B) = \left[0; \frac{2(BA^{-1})^n A\left(\frac{A-B}{A+B}\right)}{I}, \frac{-k^2 A\left(\frac{A-B}{A+B}\right)^2 A^{-1}}{(2k+1)I}\right]_{k=1}^{+\infty}.$$

Proof In order to prove this Theorem, we recall a continued fraction expansion of the relative operator entropy $S(A \mid B)$ (see [14]).

$$S(A \mid B) = \left[0; \frac{2A\left(\frac{A-B}{A+B}\right)}{I}, \frac{-k^2 A\left(\frac{A-B}{A+B}\right)^2 A^{-1}}{(2k+1)I}\right]_{k=1}^{+\infty}.$$

By applying Lemma 11 we complete the proof of Corollary. □

3.3 Representation of the operator divergence

The Bregman operator divergence is $D(A \mid B) = B - A - S(A \mid B)$. We see that its positivity is assured by

$$S(A \mid B) = A^{1/2} \ln(A^{-1/2}BA^{-1/2})A^{1/2} \leq A^{1/2}(A^{-1/2}BA^{-1/2} - I)A^{1/2} = B - A.$$

In Isa et al. [7] have generalized $D(A \mid B)$ as follows

$$D_q(A \mid B) = f_{-q}(B, A) - f_q(A, B) - S_q(A \mid B).$$

where

$$f_q(A,B) = A^{1/2}(A^{-1/2}B(A^{-1/2})^q A^{1/2}.$$

Theorem 13 *Let A and B be two positive definite matrices in $\mathcal{M}_m$, q a positive real number such that $0 < q < 1$. A continued fraction representation of the operator divergence $D_q(A \mod B)$ is given by*

$$D_q(A \mid B) = \left[B - A; \frac{(A_1' + I + \widetilde{A^*}_0\widetilde{A^*}_1)A^{1/2}}{A_1'\widetilde{A^*}_1 A^{-1/2}}, \frac{A_1'\widetilde{A^*}_1 F_2^* A^{-1/2}}{E_2^* - F_2^*}, \frac{E_{n-1}^* F_n^*}{E_n^* - F_n^*} \right]_{n=3}^{+\infty}$$

For $k \geq 0$, ${A'}_k$ are the same as in Theorem 3.3. The matrices $\widetilde{A}_k^*, E_k^*$ and F_k^* for all $k \geq 1$ are defined by the substitution of $\widetilde{A}_k'$ by b $\widetilde{A}^*$ in Theorem 3.3. with $\widetilde{A}_0^* = A^{-1/2}BA^{-1/2} - I$ and $\widetilde{A}_k^* = -\widetilde{A}_k'$ for all $k \geq 1$.

Proof Since $f_{-q}(B,A) = f_{1+q}(A,B)$, then we have

$$\begin{aligned} D_q(A \mid B) &= A^{1/2}((A^{-1/2}BA^{-1/2})^{q+1} - (A^{-1/2}BA^{-1/2})^q - \\ &(A^{-1/2}BA^{-1/2})^q ln(A^{-1/2}BA^{-1/2}))A^{1/2} \\ &= A^{1/2}(A^{-1/2}BA^{-1/2})^q(-ln(A^{-1/2}BA^{-1/2}) + \\ &A^{-1/2}BA^{-1/2} - I)A^{1/2} \end{aligned}$$

So, the proof of Theorem 13 is shown by the similar way of Theorem 10 by replacing $\ln(A^{-1/2}BA^{-1/2})$ by $(-\ln(A^{-1/2}BA^{-1/2}) + A^{-1/2}BA^{-1/2} - I)$. □

4 Numerical applications

This paragraph will provide some numerical data to illustrate the preceding results. The focus will be on the results obtained for the generalized operator entropy and the operator divergence.

4.1 Numerical example of the $A^q \ln(A)$,

We start this section by giving an example to illustrate the theoretical results obtained in theorem 3.2.

Example 1

λ	q	$\lambda^q \ln\lambda - F_1$	$\lambda^q \ln\lambda - F_2$	$\lambda^q \ln\lambda - F_3$	$\lambda^q \ln\lambda - F_4$	$\lambda^q \ln\lambda - F_5$
2	$\frac{1}{2}$	0.04692481	$0.1478\ 10^{-2}$	$0.4462\ 10^{-4}$	$0.1331\ 10^{-5}$	$0.394\ 10^{-7}$
2	$\frac{1}{3}$	0.03997739	$0.1269\ 10^{-2}$	$0.3845\ 10^{-4}$	$0.115\ 10^{-5}$	$0.343\ 10^{-7}$
3	$\frac{1}{2}$	0.2361856	0.01855478	$0.1369\ 10^{-2}$	$0.9973\ 10^{-4}$	$0.7224\ 10^{-5}$

Now, we pass to a more general case than the previous one; matrix case.

Example 2 Let $A \in \mathcal{M}_m$ be a positive definite matrix, such that

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

We calculate the difference between the exact value of $A^q lnA$ and its first convergent. We will take for example $q = \frac{1}{3}$, we get

$$A^q lnA - F_1 = \begin{pmatrix} 0.08867877674 & 0.1379741093 \\ 0.1379741091 & 0.226652885 \end{pmatrix}$$

$$A^q lnA - F_2 = \begin{pmatrix} 0.00934803094 & 0.0150871241 \\ 0.0150871239 & 0.024435154 \end{pmatrix}$$

$$A^q lnA - F_3 = \begin{pmatrix} 0.93618974\, 10^{-3} & 0.15145295\, 10^{-2} \\ 0.15145293\, 10^{-2} & 0.2450718\, 10^{-2} \end{pmatrix}$$

$$A^q lnA - F_4 = \begin{pmatrix} 0.9212614\, 10^{-4} & 0.1490607\, 10^{-3} \\ 0.1490605\, 10^{-3} & 0.241186\, 10^{-3} \end{pmatrix}$$

$$A^q lnA - F_5 = \begin{pmatrix} 0.900254\, 10^{-5} & 0.145655\, 10^{-4} \\ 0.145653\, 10^{-4} & 0.23567\, 10^{-4} \end{pmatrix}$$

4.2 Numerical example of the generalized operator entropy

Now, we turn to the first main objective of this work; approximation of the generalized operator entropy by using continued fractions

Example 3 Let A and B be two positive definite matrices in $\mathcal{M}_m$ such that

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

We calculate the difference between $S_q(A \mid B)$ and its first five convergents, we will take for example $q = \frac{1}{3}$, we got the following results

$$F_1 - S_q(A \mid B) = \begin{pmatrix} 0.054190989 & -0.04186083842 & 0.0061362095 \\ -0.0418608386 & -0.09860205655 & -0.0089742420 \\ 0.00613620937 & -0.0089742424 & 0.041304948 \end{pmatrix}$$

$$F_2 - S_q(A \mid B) = \begin{pmatrix} 0.001377942 & -0.00151980232 & 0.0002817516 \\ -0.0015198025 & -0.00384514555 & -0.0004732079 \\ 0.00028175148 & -0.0004732083 & 0.000786263 \end{pmatrix}$$

$$F_3 - S_q(A \mid B) = \begin{pmatrix} 0.000032775 & -0.00005387752 & 0.86254\,10^{-5} \\ -0.0000538777 & -0.00014206055 & -0.0000193084 \\ 0.862533\,10^{-5} & -0.0000193088 & 0.000014661 \end{pmatrix}$$

$$F_4 - S_q(A \mid B) = \begin{pmatrix} 0.718\,10^{-6} & -0.190962\,10^{-5} & 0.2162\,10^{-6} \\ -0.19098\,10^{-5} & -0.516755\,10^{-5} & -0.7279\,10^{-6} \\ 0.21615\,10^{-6} & -0.7283\,10^{-6} & 0.263\,10^{-6} \end{pmatrix}$$

$$F_5 - S_q(A \mid B) = \begin{pmatrix} 0.13\,10^{-7} & -0.6722\,10^{-7} & 0.65\,10^{-8} \\ -0.674\,10^{-7} & -0.18655\,10^{-6} & -0.272\,10^{-7} \\ 0.643\,10^{-8} & -0.276\,10^{-7} & -0.2\,10^{-8} \end{pmatrix}$$

4.3 Numerical example of the operator divergence

We end this paragraph by illustrating the theoretical results concerning the divergence operator.

We keep the same data as we have in the previous example. We calculate the difference between $D_q(A \mid B)$ and its first five convergents, we got the following results

Example 4

$$F_1 - D_q(A \mid B) = \begin{pmatrix} -0.069264959 & 0.03318154762 & -0.0099888506 \\ 0.0331815487 & 0.07019678255 & 0.0044140547 \\ -0.00998885042 & 0.0044140552 & -0.048288369 \end{pmatrix}$$

$$F_2 - D_q(A \mid B) = \begin{pmatrix} -0.001821578 & 0.00117926062 & -0.0004227273 \\ 0.0011792617 & 0.00278467055 & 0.0003187543 \\ -0.00042272712 & 0.0003187548 & -0.000933848 \end{pmatrix}$$

$$F_3 - D_q(A \mid B) = \begin{pmatrix} -0.000045885 & 0.00004116862 & -0.0000132690 \\ 0.0000411697 & 0.00010356255 & 0.0000139272 \\ -0.00001326882 & 0.0000139277 & -0.000018017 \end{pmatrix}$$

$$F_4 - D_q(A \mid B) = \begin{pmatrix} -0.1115\,10^{-5} & 0.144362\,10^{-5} & -0.3629\,10^{-6} \\ 0.14447\,10^{-5} & 0.378555\,10^{-5} & 0.5380\,10^{-6} \\ -0.36272\,10^{-6} & 0.5385\,10^{-6} & -0.350\,10^{-6} \end{pmatrix}$$

$$F_5 - D_q(A \mid B) = \begin{pmatrix} -0.27\,10^{-7} & 0.5062\,10^{-7} & -0.100\,10^{-7} \\ 0.507\,10^{-7} & 0.14255\,10^{-6} & 0.217\,10^{-7} \\ -0.982\,10^{-8} & 0.222\,10^{-7} & -0.3\,10^{-8} \end{pmatrix}$$

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