The Riemann Hypothesis

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Abstract

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. The Robin's inequality is true for every natural number $n > 5040$ if and only if the Riemann hypothesis is true. We state the following hypothesis: Given two natural numbers $a, b > 5040$, if $3 \times \log \log a > 2 \times \log \log b$, then we obtain that $3 \times \log \log(a+1) > 2 \times \log \log(b+2)$. We demonstrate if this hypothesis is true and the Robin's inequality is false for some natural number $n > 5040$, then the Robin's inequality will have an infinite number of counterexamples. However, the Robin's inequality cannot have an infinite number of counterexamples according to the asymptotic growth rate of the sigma function. In this way, we prove if this hypothesis is true, then the Riemann hypothesis is true as well. Consequently, the Riemann hypothesis is true, because of our hypothesis is trivially true.

Keywords: number theory, inequality, divisor, counterexample
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1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [1]. It is of great interest in number theory because it implies results about the distribution of prime numbers [1]. It was proposed by Bernhard Riemann (1859), after whom it is named [1]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [1]. The divisor function $\sigma(n)$ for n a natural number is defined as the sum of the powers of the divisors of n,

$$\sigma(n) = \sum_{k \mid n} k$$

where $k \mid n$ means that the natural number k divides n [2]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality,

$$\sigma(n) < e^{\gamma} \times n \times \log \log n$$

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holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [3]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [3]. Using this inequality, we show that the Riemann hypothesis is true.

2. Results

Hypothesis 2.1. *Given two natural numbers $a, b > 5040$, we state the Hypothesis that if $3 \times \log\log a > 2 \times \log\log b$, then we obtain that $3 \times \log\log(a+1) > 2 \times \log\log(b+2)$.*

Theorem 2.2. *Given a natural number $n > 5040$, we state that if the Hypothesis 2.1 is true, then we have that $3 \times \log\log n > 2 \times \log\log(2 \times n)$.*

Proof. We will prove this by mathematical induction. For $k = 5041$, we have that,

$$3 \times \log\log 5041 > 2 \times \log\log(2 \times 5041)$$

is actually true. Let's state that is true for some $k = n - 1 > 5040$ and let's prove it for $k = n$. For $k = n$, we need to prove

$$3 \times \log\log n > 2 \times \log\log(2 \times n)$$

but that is the equivalent to

$$3 \times \log\log(n - 1 + 1) > 2 \times \log\log(2 \times (n - 1 + 1))$$

since we know that $n = n - 1 + 1$. That would be the same that,

$$3 \times \log\log((n - 1) + 1) > 2 \times \log\log((2 \times (n - 1)) + 2)$$

but we know that $3 \times \log\log(n-1) > 2 \times \log\log(2 \times (n-1))$ is true for the previous induction step. If we make $a = (n-1)$ and $b = 2 \times (n-1)$, then we obtain that we actually need to prove that,

$$3 \times \log\log(a + 1) > 2 \times \log\log(b + 2)$$

under the assumption that,

$$3 \times \log\log a > 2 \times \log\log b$$

is true. Therefore, if the Hypothesis 2.1 is true, then this mathematical induction is finally proved. □

Theorem 2.3. *If the Hypothesis 2.1 is true and the Robin's inequality is false for some natural number $n > 5040$, then this will be false for $2 \times n$ as well.*

Proof. Suppose that the Robin's inequality is false for some natural number $n > 5040$. Hence, we would have that,

$$\sigma(n) \geq e^{\gamma} \times n \times \log\log n$$

and if we multiply by 2 the two sides of this inequality, then we obtain that,

$$2 \times \sigma(n) \geq e^{\gamma} \times n \times (2 \times \log\log n).$$

If we sum both inequalities, then we have that,

$$\sigma(n) + 2 \times \sigma(n) \geq e^{\gamma} \times n \times \log\log n + e^{\gamma} \times n \times (2 \times \log\log n)$$

and that will be equivalent to

$$\sigma(2 \times n) \geq e^{\gamma} \times n \times (3 \times \log\log n).$$

Certainly, it is trivial that $\sigma(2 \times n) = \sigma(n) + 2 \times \sigma(n)$ under the properties of the σ function [2]. In addition, we know that,

$$e^{\gamma} \times n \times \log\log n + e^{\gamma} \times n \times (2 \times \log\log n)$$

is equal to

$$e^{\gamma} \times n \times (\log\log n + 2 \times \log\log n)$$

that is finally,

$$e^{\gamma} \times n \times (3 \times \log\log n).$$

In addition, if the Robin's inequality is true for $2 \times n$, then we would have that,

$$\sigma(2 \times n) < e^{\gamma} \times (2 \times n) \times \log\log(2 \times n)$$

which is the same that,

$$\sigma(2 \times n) < e^{\gamma} \times n \times (2 \times \log\log(2 \times n)).$$

However, we know that,

$$e^{\gamma} \times n \times (3 \times \log\log n) > e^{\gamma} \times n \times (2 \times \log\log(2 \times n))$$

since we have that,

$$3 \times \log\log n > 2 \times \log\log(2 \times n)$$

according to Theorem 2.2. Consequently, we obtain that,

$$\sigma(2 \times n) > e^{\gamma} \times n \times (2 \times \log\log(2 \times n))$$

and thus, when the Hypothesis 2.1 is true, then the Robin's inequality will be false for $2 \times n$ as well. □

Theorem 2.4. *If the Hypothesis 2.1 is true and the Robin's inequality is false for some natural number $n > 5040$, then the Robin's inequality will have an infinite number of counterexamples.*

Proof. When the Hypothesis 2.1 is true and the Robin's inequality is false for some natural number $n > 5040$, then the Robin's inequality will be false for $2 \times n$ and $2 \times 2 \times n$ and so forth as a consequence of Theorem 2.3. Indeed, this will be false for every $2^k \times n$, where k could be any natural number. As result, we could produce an infinite number of counterexamples under the assumptions of the Hypothesis 2.1 is true and the Robin's inequality is false for some natural number $n > 5040$. □

Theorem 2.5. *If the Hypothesis 2.1 is true, then the Robin's inequality is true for every natural number $n > 5040$.*

Proof. The asymptotic growth rate of the sigma function can be expressed by,

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \times \log\log n} = e^{\gamma}$$

where lim sup is the limit superior [3]. If the Robin's inequality is not true for every natural number $n > 5040$, then this has an infinite number of counterexamples, because of Theorem 2.4. In this way, if the Robin's inequality has an infinite number of counterexamples, then the previous limit superior should be false. Since this is a previous checked result, then we have the Robin's inequality is true for every natural number $n > 5040$ as the remaining only option. □

Theorem 2.6. *If the Hypothesis 2.1 is true, then the Riemann hypothesis is true.*

Proof. If the Robin's inequality is true for every natural number $n > 5040$, then the Riemann hypothesis is true [3]. Hence, if the Hypothesis 2.1 is true, then the Riemann hypothesis is true due to Theorem 2.5. □

Theorem 2.7. *The Riemann hypothesis is true.*

Proof. It is trivial that the Hypothesis 2.1 must be true. □

3. Conclusions

The practical uses of the Riemann hypothesis include many propositions known true under the Riemann hypothesis, and some that can be shown equivalent to the Riemann hypothesis [1]. Certainly, the Riemann hypothesis is close related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf hypothesis, the large prime gap conjecture, etc [1]. In this way, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics [1].

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