Many-Fold Fuzzy Semantics of Many-Place Sequent Calculi with Enlargement

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ABSTRACT. In this paper we propose and study a semantics of many-place sequent calcul with Enlargement as well as implicit Permutation and Contraction based upon the conception of *many-fold fuzzy set* being a natural extension of that of *two-fold* one. In the propositional case, we come to the conception of fuzzy many-place matrix that is a fuzzification of the conception of many-place matrix proposed by us in earlier papers and provides semantics of propositional calculi of the kind involved.

1. INTRODUCTION

Since appearance of the conception of *fuzzy set* [60], its applications to various branches of Mathematics and Computer Science have become more than miscellaneous and, in many cases, even rather unexpected. Just recently, the idea of "L-fuzzyfication" (cf. [11]) has been used in [27]/[30] [and advanced in [37]]/ for providing semantics of derivable rules [schemas] going back to [20, 21] (instead of that of merely derivable axioms, following the paradigm of [23, 24, 29, 33, 32, 34, 44, 45, 47, 46, 48]) of two-side (viz., ordinary Gentzen-style; cf. [10]) multiple-conclusion sequent calculi with structural/"weak (viz., ortho-)structural" rules upon the basis of L-fuzzyfication of the notions of ordinary (viz., bi-valential) valuation and interpretation of sequents in it (in its turn, going back to [24, 29, 32]) based upon natural treatment of two-side propositional sequents as *clauses* (cf. [51]) of the first-order signature with single unary *assertion/truth* predicate. It is remarkable that, therein, fuzzyfication has been involved to study crisp objects — sequent calculi, in their turn, having substantial applications to Automated Reasoning (cf. [9])[1] and, more generally, to such advanced branches of Computer Science as Artificial Intelligence, especially when involving minimality/optimality issues like in [42, 43, 40] as well as those of either program implementation like in [31, 35, 36, 41] or many-sorted framework (like in [48]) going back to [37].

Nevertheless, the universal framework of [27] and [30] has proved, in principle, too restrictive to cover the following generic classes of sequent calculi:

(1) any kind of many-place sequent calculi (cf. [52], [53]) including Tait-syle (viz., one-place; cf. [59]) calculi — viz., *signed* sequent calculi according to the equivalent *signed sequent* formalism/paradigm going back to [57];
(2) two-side sequent calculi without Cut and/or Sharing;

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[1] In this connection, recall that the Sequent approach to Automated Deduction, though being equivalent to the Resolution one [51, 50] within the context of classical logic, is equally applicable to paraconsistent (more generaly, relevance) logics (such as [1, 58]), while the Resolution rule (more precisely, its instance — the *Ex Contradictione Quodlibet* one — is not derivable in them.

The primary goal of the present paper is to cover these two classes. It appears that this is coherently fulfilled by means of involving the conception of *many-fold* L-fuzzy set, being a tuple of L-fuzzy sets with same basic set and grading lattice, that naturally extends the conception of *two-fold* fuzzy set [6, 7], being sufficient for the second class. And what is more, involving either Cut or Sharing makes two-fold fuzzy sets under consideration couples of measures of necessity and possibility, as in [6, 7, 8]. In that case, results of [27] become particular cases of those to be proved below but with essentially different argumentation based upon generic advanced results, some of which actually extend those of [32] and [39] to non-propositional case. Among other things, new argumentation discloses atomic Booleanity of L-fuzzy sets involved in addition to their lattice completeness implicitly discovered in [27].

Throughout the paper, we mainly follow the formalism of [27] and [30] except that, for simplifying the overall exposition, sequent places (that is, sides in the two-side case) are treated here as rather finite sets than finite sequences of formulas, in which case Permutation and Contraction become trivial rules and, for this reason, are not considered at all.

The rest of the paper is as follows. In Section 2, we mainly specify basic notions and notations and argue some underlying issues to be used further. Section 3 incorporates main generic results concerning sequent calculi. In Section 4, we exemplify our general elaboration by studying a cut and/or sharing-free (and so being beyond the scopes of [27] at all) multiplicative two-side sequent calculi resulted from Gentzen's calculus [10] by adding rules inverse to logical ones, the empty-sequent-less fragment of the former having been studied in [38], as well as two multiple-conclusion Gentzen-style axiomatizations of FDE [3] going back to [21, 23, 24]. Finally, Section 5 is a concise summary of principal contributions of the paper and a brief outline of further related work.

2. General background

Unless otherwise specified, we entirely follow standard conventions concerning Set and Lattice Theory as well as Universal Algebra to be found, e.g., in [2, 4, 5, 12, 13, 14, 17, 9, 56]. In addition, abstract algebras are denoted by Fraktur letters [possibly, with indices], their carriers (viz., underlying sets) being denoted by corresponding Italic letters [with same indices, if any].

2.1. **Set- and lattice-theoretic preliminaries.**

We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser ones), the countable/proper ordinal/class of all them/ordinals being denoted by ω/∞.

Likewise, functions are viewed as binary relations. In addition, singletons are often identified with their unique elements, unless any confusion is possible.

Given a set S, the set of all subsets of S [of cardinality in any class $K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$. Then, for any $A \subseteq S$, we have its *characteristic function* $\chi^A_S \triangleq ((A \times \{1\}) \cup ((S \setminus A) \times \{0\})) : S \to 2$. Next, S-tuples (viz., functions with domain S) are normally written in either sequence $\bar{t}$ or vector $\vec{t}$ forms, its s-th component, where $s \in S$, being written as t_s, in that case. Likewise, elements of $S^{*/+} \triangleq \bigcup_{i\in(\omega\setminus(0/1))} A^i$ are identified with ordinary finite /non-empty tuples or sequences, the binary concatenation operation on S^* being denoted by $*$, as usual. Further, set $\Delta_S \triangleq \{\langle a, a\rangle | a \in S\}$, binary relations of such a kind being said to be *diagonal*. Any binary operation $\diamond$ on S /"collectively with any $b \in S$" determines the mapping $\diamond^{/b} : S^{+/*} \to S$ as follows: by induction on the length $l = (\operatorname{dom} \bar{a})$ of any $\bar{a} \in S^{+/*}$, put:

$$\diamond^{/b}\bar{a} \triangleq \begin{cases} a_0/b & \text{if } l = (1/0), \\ (\diamond^{/b}(\bar{a} \upharpoonright (l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

In particular, given any $f : S \to S$ and any $n \in \omega$, we have $f^n \triangleq ((\circ \upharpoonright S^S)^{\Delta_S}(n \times \{f\})) : S \to S$. Finally, given also an indexed system $\{T_j\}_{j\in J}$ of sets, any $\bar{f} \in \prod_{j\in J} T_j^S$ determines the mapping $(\prod \bar{f}) = (\prod_{j\in J} f_j) : S \to (\prod_{j\in J} T_j), a \mapsto \langle f_j(a)\rangle_{j\in J}$.

Let A be a set. Given any $S \subseteq \wp(A)$, an $M \in S$ is said to be *maximal*, provided $\{T \in S | M \subseteq T\} \subseteq \{M\}$, the set of all them being denoted by $\max(S)$. A $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$, in which case $U \neq \varnothing$, when taking $S = \varnothing$. An $S \subseteq \wp(A)$ is said to be *inductive*, provided, for every upward-directed $U \subseteq S$, it holds that $(\bigcup U) \in S$, in which case, by Zorn's Lemma (cf. [18, 9]), $(S \neq \varnothing) \Rightarrow (\max(S) \neq \varnothing)$. A *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$, in which case any $\mathcal{B} \subseteq \mathcal{C}$ is called a *closure basis of* $\mathcal{C}$, provided $\mathcal{C} = \{A \cap \bigcap S | S \subseteq \mathcal{B}\}$. An *operator over* A is any unary operation O on $\wp(A)$. This is said to be *"monotonic/idempotent/transitive"*|*"inductive/finitary"*, provided, for "all $\overline{B} \in \wp(A)^{2/1/1}$"|"any upward-directed $U \subseteq \wp(A)$", it holds that $(O^{1/0/2}(B_0) \subseteq O(B_{1/0/0}))|(O(\bigcup U) \subseteq \bigcup O[U])$. A *closure operator over* A is any monotonic idempotent transitive operator over A. Given any [inductive and] monotonic operator O over A, a $B \in \wp(A)$ is said to be O*-closed*, provided $O(B) \subseteq B$, the set $\mathrm{Cl}(O)$ of all O-closed elements of $\wp(A)$ being a[n inductive] closure system over A. Finally, for any closure operator C over A, we have $\mathrm{Cl}(C) = C[\wp(A)]$, while any $\mathcal{B} \subseteq \mathrm{Cl}(C)$ is a closure basis of $\mathrm{Cl}(C)$ iff $C(D) = (A \cap \bigcap\{B \in \mathcal{B} | D \subseteq B\})$, for all $D \in \wp(A)$.

Let $\mathcal{P} = \langle P, \leqslant^{\mathcal{P}}\rangle$ be a poset. Given any $S \subseteq P$, a/an *lower/upper bound of* S is any $a \in (P[\cap S])$ such that $a(\leqslant / \geqslant)^{\mathcal{P}} b$, for each $b \in S$ [then called the *least/greatest* element of S], the greatest/least one (if any) being denoted by $(\bigwedge / \bigvee)^{\mathcal{P}} S$ and called the *meet/join of* S. Then, $\mathcal{P}$ is referred to as a *[complete|bounded] lattice*, provided every $S \in \wp_{(\omega\setminus 1)[\cup(\infty|\omega)]}(P)$ has both meet and join. In that case, we, as usual, write $[(1/0)^{\mathcal{P}}$ for the *unit/zero* $(\bigwedge / \bigvee)^{\mathcal{P}}\varnothing$ *of* $\mathcal{P}$ and] $a(\wedge/\vee)^{\mathcal{P}} b$ for $(\bigwedge / \bigvee)^{\mathcal{P}}\{a, b\}$, where $a, b \in P$, while S is called a $\{$*prime*$\}$ *filter/ideal of* $\mathcal{P}$, whenever, for all $a, b \in P$, $((a(\wedge/\vee)^{\mathcal{P}} b) \in S) \iff (\{a, b\} \subseteq S)$ $\{$whereas $P \setminus S$ is an/a ideal/filter of $\mathcal{P}\}$, as well as $\mathcal{P}$ is referred to as [*complemented*, whenever, each $a \in P$ has a *complement in* $\mathcal{P}$ (viz., some $b \in P$ such that $(a(\wedge/\vee)^{\mathcal{P}} b) = (0/1)^{\mathcal{P}}$) and] *[completely|] distributive*, whenever for all $A \in \wp_{(\omega\setminus 1)[\cup(\infty|\omega)]}(P)$ and all $\vec{B} \in \wp_{(\omega\setminus 1)[\cup(\infty|\omega)]}(P)^A$, it holds that $(\bigwedge / \bigvee)^{\mathcal{P}}_{a\in A}(\bigvee / \bigwedge)^{\mathcal{P}} B_a = (\bigvee / \bigwedge)^{\mathcal{P}}_{f\in\prod \vec{B}}(\bigwedge / \bigvee)^{\mathcal{P}}_{a\in A} f(a)$. In general, any mention of $\mathcal{P}$ (including the superscript) is often omitted, unless any confusion is possible. Likewise, as usual, "Boolean" stands for "complemented distributive bounded". We equally follow the conventional [infinitary|] algebraic representation of [complete|bounded] lattices (cf., e.g., [2, 12, 13, 55, 56]) tacitly, according to which, in particular, direct products of indexed families of them are defined in the standard algebraic manner (i.e., by setting $\prod_{i\in I}\langle P_i, \leqslant^{\mathcal{P}_i}\rangle \triangleq \langle \prod_{i\in I} P_i, \{\langle \bar{a}, \bar{b}\rangle \in (\prod_{i\in I} P_i)^2 \mid \forall i \in I : a_i \leqslant^{\mathcal{P}_i} b_i\}\rangle$). Given any set S, $\langle \wp(S), \subseteq \cap\, \wp(S)^2\rangle$ is a complemented completely distributive complete lattice called the *power* one *of* S and identified with $\wp(S)$, in which case any two-element one is isomorphic to $\wp(1) = 2$, and so [finite] ones are exactly isomorphic copies of direct [finite] powers of 2 (cf. [55]).

2.2. **Formal languages and calculi.** A *(formal) language* is a couple of the form $L = \langle \mathrm{Fm}_L, \mathrm{Sb}_L\rangle$, where Fm_L is a set, whose elements are referred to as L*-formulas*, and Sb_L is a set of unary operations on Fm_L closed under composition and containing $\iota_L \triangleq \Delta_{\mathrm{Fm}_L}$, whose elements are referred to as *L-substitutions*.

Elements of $\mathrm{Ru}_L^{[\omega]} \triangleq (\wp_{[\omega]}(\mathrm{Fm}_L) \times \mathrm{Fm}_L)$ are referred to as *[finitary] L-rules*, any $\langle \Gamma, \Phi \rangle \in \mathrm{Ru}_L$ being normally written in either of conventional forms $\Gamma \to \Phi$ or $\frac{\Gamma}{\Phi}$, Φ/"elements of Γ" being referred to as its *conclusion/premises*, L-rules of the form $\Psi \to \Phi$, where $\Psi \in \Gamma$, being referred to as *inverse to* $\Gamma \to \Phi$, L-rules of the form $\sigma(\Gamma \to \Phi) \triangleq (\sigma[\Gamma] \to \sigma(\Phi))$, where $\sigma \in \mathrm{Sb}_L$, being referred to as *(substitutional) L-instances of* $\Gamma \to \Phi$. L-Rules with(out) premises are said to be *proper* or *non-axiomatic* (resp., called *L-axioms* and identified with their conclusions). Rules with conclusion being one of premises are said to be *trivial*.

An *L-calculus* is any $\mathbb{C} \subseteq \mathrm{Ru}_L$, in which case we set $(\mathbb{C} \upharpoonright \omega) \triangleq (\mathbb{C} \cap \mathrm{Ru}_L^{[\omega]})$. Then, $\mathbb{C}$ is said to be *finitary*, whenever $(\mathbb{C} \upharpoonright \omega) = \mathbb{C}$. Further, $\mathbb{C}$ is said to be *schematic*, provided it contains every L-instance of each of its elements.

An *L-valuation* is any $v \subseteq \mathrm{Fm}_L$, in which case $(\Gamma \to \Phi) \in \mathrm{Ru}_L$ is said to be *true/valid/satisfied in v under* $\sigma \in \mathrm{Sb}_L$ ($v \models (\Gamma \to \Phi)[\sigma]$, in symbols), if $(\sigma[\Gamma] \subseteq v) \Rightarrow (\sigma(\Phi) \in v)$, and *true/valid/satisfied in* v, ($v \models (\Gamma \to \Phi)$, in symbols), whenever it is true in v under each L-substitution. Next, v is said to be *total/proper*, if $v = / \neq \mathrm{Fm}_L$. Further, we also have the L-valuation $\complement_L(v) \triangleq (\mathrm{Fm}_L \setminus v)$ said to be *complementary to* v, in which case $\complement_L(\complement_L(v)) = v$. Finally, given any L-calculus $\mathbb{C}$, the class of all[proper] L-valuations satisfying each member of $\mathbb{C}$ is denoted by $\mathrm{Val}^{[*]}(\mathbb{C})$.

An *L-consequence (relation)* is any L-calculus $\vdash$ satisfying the following consequence conditions:

$$\begin{array}{ll} \text{(Reflexivity)} & \Phi \vdash \Phi, \\ \text{(Monotonicity)} & (\Gamma \vdash \Phi \,\&\, \Gamma \subseteq \Xi) \Rightarrow \Xi \vdash \Phi, \\ \text{(Transitivity)} & (\Gamma \vdash \Xi \,\&\, \Xi \vdash \Phi) \Rightarrow \Gamma \vdash \Phi, \end{array}$$

for all $\Gamma, \Xi \in \wp(\mathrm{Fm}_L)$ and $\Phi \in \mathrm{Fm}_L$. (We adopt the following natural abbreviations: $\Gamma \vdash \Phi$ is used for $(\Gamma \to \Phi) \in \vdash$, $\Gamma \vdash \Xi$ means $\forall \Psi \in \Xi : \Gamma \vdash \Psi$.) Further, we have the L-consequence relation $\vdash^\omega \subseteq \vdash$ defined as follows: for all $(\Gamma \to \Phi) \in \mathrm{Ru}_L$, set:

$$(\Gamma \vdash^\omega \Phi) \iff \exists \Xi \in \wp_\omega(\Gamma) : (\Xi \vdash \Phi).$$

(Notice that $\vdash^\omega$ is schematic, whenever $\vdash$ is so.) Then, $\vdash$ is said to be *compact*, provided $\vdash \subseteq \vdash^\omega$. Further, $\vdash$ is said to be *[in]consistent*, if it is [not] distinct from Ru_L. In view of the reflexivity, monotonicity and transitivity of $\vdash$, we have the closure operator $\mathrm{Cn}_\vdash$ over Fm_L, defined by $\mathrm{Cn}_\vdash(\Gamma) \triangleq \{\Phi \in \mathrm{Fm}_L \mid \Gamma \vdash \Phi\}$, for all $\Gamma \subseteq \mathrm{Fm}_L$. (Note that $\mathrm{Cn}_\vdash$ is inductive iff $\vdash$ is compact.) It is routine checking that:

$$\mathrm{Val}(\vdash) = \mathrm{Cl}(\mathrm{Cn}_\vdash), \tag{1}$$

provided $\vdash$ is schematic.

An *L-semantics* is any set S of L-valuations. In that case, the set $\vdash_{\mathsf{S}}$ of all L-rules *true in* S, that is, true in each member of it, is a schematic L-consequence (for Sb_L is closed under composition), said to be the *semantic* one *of* or *defined by* S. As the consequence of the total L-valuation is inconsistent, we have:

$$\vdash_{\mathsf{S}} = \vdash_{\mathrm{Pr}(\mathsf{S})}, \tag{2}$$

where $\mathrm{Pr}(\mathsf{S})$ denotes the class of all proper members of S. Further, set:

$$\begin{aligned} \mathbf{S}^{-1}(\mathsf{S}) &\triangleq \{\sigma^{-1}[v] \mid v \in \mathsf{S}, \sigma \in \mathrm{Sb}_L\}, \\ \mathbf{M}^{[*]}(\mathsf{S}) &\triangleq \{\mathrm{Fm}_L \cap \bigcap S \mid [\varnothing \neq] S \subseteq \mathsf{S}\}. \end{aligned}$$

Lemma 2.1. *Any schematic L-consequence $\vdash$ is defined by any closure basis $\mathcal{B}$ of* $\mathrm{Cl}(\mathrm{Cn}_{\vdash})$ *(in particular, by* $\mathrm{Val}(\vdash)$*).*

Proof. Consider any $(\Gamma \to \Phi) \in (\mathrm{Ru}_L \setminus \vdash)$. Then, $\Phi \notin \mathrm{Cn}_{\vdash}(\Gamma)$, in which case there is some $v \in \mathcal{B} \subseteq \mathrm{Cl}(\mathrm{Cn}_{\vdash})$ such that $\Gamma \to \Phi$ is not true in v under ι_L. Then, (1) completes the argument. ∎

Let $\mathbb{C}$ be an L-calculus. An *extension of* $\mathbb{C}$ is any schematic L-consequence including $\mathbb{C}$. The least extension of $\mathbb{C}$ is denoted by $\vdash_{\mathbb{C}}$ and said to be the *consequence of* (or *axiomatized by*) $\mathbb{C}$.[2] (When $\mathbb{C}$ is a schematic L-consequence, we simply have $\vdash_{\mathbb{C}} = \mathbb{C}$; in particular, $\vdash_{\vdash_{\mathbb{C}}} = \vdash_{\mathbb{C}}$, in any case.) In case $\mathbb{C}$ is finitary, $\vdash^{\omega}_{\mathbb{C}}$ is an extension of $\mathbb{C}$, and so $\vdash_{\mathbb{C}}$ is compact. Conversely, given any schematic L-consequence $\vdash$, $\vdash^{\omega}$ is axiomatized by $\vdash \upharpoonright \omega$, in which case any compact schematic L-consequence is axiomatized by a finitary L-calculus. An extension of $\mathbb{C}$ is said to be *relatively axiomatized by* an L-calculus $\mathbb{R}$, provided it is axiomatized by $\mathbb{C} \cup \mathbb{R}$ or, equivalently, by $\vdash_{\mathbb{C}} \cup \mathbb{R}$, and *finitary*, whenever it is relatively axiomatized by a finitary L-calculus.

Lemma 2.2. *Let $\mathbb{C}$ be an L-calculus. Then,* $\mathrm{Val}(\mathbb{C})$ *is closed under* $\mathbf{S}^{-1}$ *and* $\mathbf{M}$.

Proof. Notice that an L-valuation v belongs to $\mathrm{Val}(\mathbb{C})$ iff, for every $\sigma \in \mathrm{Sb}_L$, it holds that $\sigma^{-1}[v] \in \mathrm{Cl}(\mathrm{Cn}_{\mathbb{C}})$.

Consider any $\sigma \in \mathrm{Sb}_L$. Then, for any $S \subseteq \wp(\mathrm{Fm}_L)$, we have $\sigma^{-1}[\bigcap S] = \bigcap\{\sigma^{-1}[T] | T \in S\}$, so $\mathrm{Val}(\mathbb{C})$ is closed under $\mathbf{M}$. Moreover, for any $\sigma' \in \mathrm{Sb}_L$, we have both $(\sigma \circ \sigma') \in \mathrm{Sb}_L$ and $(\sigma \circ \sigma')^{-1} = (\sigma'^{-1} \circ \sigma^{-1})$, so $\mathrm{Val}(\mathbb{C})$ is closed under $\mathbf{S}^{-1}$, as required. ∎

Theorem 2.3. *Let S be an L-semantics. Then,*

$$\mathrm{Val}^{[*]}(\vdash_{\mathsf{S}}) = \mathbf{M}^{[*]}([\mathrm{Pr}(]\mathbf{S}^{-1}([\mathrm{Pr}(]\mathsf{S}[)])[)]).$$

Proof. The inclusion from right to left is by Lemma 2.2 and the inclusion $\mathsf{S} \subseteq \mathrm{Val}(\vdash_{\mathsf{S}})$.

Conversely, in view of Lemma 2.2 and (1), $\mathbf{S}^{-1}(\mathsf{S}) \subseteq \mathrm{Cl}(\mathrm{Cn}_{\mathsf{S}})$ is a closure basis of $\mathrm{Cl}(\mathrm{Cn}_{\mathsf{S}})$. In this way, (1) completes the argument of the non-optional case[, from which the optional one ensues immediately]. ∎

Let $\mathbb{C}$ be a finitary L-calculus. Given any $\Gamma \subseteq \mathrm{Fm}_L$ [and any $\Phi \in \mathrm{Fm}_L$], a $\mathbb{C}$-*derivation [of Φ] from* Γ is any $\partial \in \mathrm{Fm}_L^*$ [with $\Phi \in (\mathrm{img}\,\partial)$] such that, for every $k \in (\mathrm{dom}\,\partial)$, either $\partial(k) \in (\Gamma \cup \partial[k])$ or there is some L-instance $\Xi \to \Psi$ of a rule in $\mathbb{C}$ such that $\Psi = \partial(k)$, whereas $\partial \upharpoonright k$ is a $\mathbb{C}$-derivation of each element of Ξ from Γ [in which case Φ is said to be *derivable in* $\mathbb{C}$ *from* Γ]. An L-rule $\Gamma \to \Phi$ is said to be *derivable in* $\mathbb{C}$, whenever Φ is derivable in $\mathbb{C}$ from Γ. The set of all L-rules derivable in $\mathbb{C}$ is denoted by $\Vdash_{\mathbb{C}}$.

Proposition 2.4. *Let $\mathbb{C}$ be a finitary L-calculus. Then,* $\vdash_{\mathbb{C}} = \Vdash_{\mathbb{C}}$.

Proof. The reflexivity of $\Vdash_{\mathbb{C}}$ is by taking $\partial = \{\langle 0, \Phi\rangle\}$, where $\Phi \in \mathrm{Fm}_L$. The monotonicity is evident. For proving the transitivity, consider any $\Gamma, \Delta \in \wp(\mathrm{Fm}_L)$, any $\Phi \in \mathrm{Fm}_L$, any $\mathbb{C}$-derivations ∂ of Φ from Δ and ∂_{Ψ} of each $\Psi \in \Delta$ from Γ. Then, ∂ is a $\mathbb{C}$-derivation of Φ from $\Xi = (\Delta \cap (\mathrm{img}\,\partial)) \in \wp_{\omega}(\Delta)$, in which case there is some bijection $\vec{\Upsilon}$ from $n \triangleq |\Xi| \in \omega$ onto Ξ, and so $*\langle\langle \partial_{\Upsilon_i}\rangle_{i \in n}, \partial\rangle$ is a $\mathbb{C}$-derivation of Φ from Γ. Thus, $\Vdash_{\mathbb{C}}$ is an L-consequence. Next, for any $\sigma \in \mathrm{Sb}_L$ and any $\mathbb{C}$-derivation ∂ from any $\Gamma \subseteq \mathrm{Fm}_L$, $\partial \circ \sigma$ is a $\mathbb{C}$-derivation from $\sigma[\Gamma]$, for Sb_L is closed under $\circ$, so $\Vdash_{\mathbb{C}}$ is schematic. Further, for any $(\Gamma \to \Phi) \in \mathbb{C}$, there is

[2] When dealing with scripts (e.g., those of Cn or $\vdash$), we normally write O for $\vdash_O$, where O is either $\mathbb{C}$ or S.

some bijection $\vec{\Omega}$ from $|\Gamma| \in \omega$ onto Γ, in which case $\langle\vec{\Omega}, \Phi\rangle$ is a $\mathbb{C}$-derivation of Φ from Γ, for $\iota_L \in \mathrm{Sb}_L$, and so $\Vdash_{\mathbb{C}}$ is an extension of $\mathbb{C}$. Finally, by induction on the length of $\mathbb{C}$-derivations, it is routine checking that $\Vdash_{\mathbb{C}} \subseteq \vdash$, for any extension $\vdash$ of $\mathbb{C}$, as required. ∎

Proposition 2.4 is often used tacitly throughout the rest of the paper. The issue of derivation is equally applicable to infinitary calculi but with involving the apparatus of transfinite ordinal arithmetics (cf. [18]). However, such an extension would be no matter for arguing main results of the paper. For this reason, we have refrained from redundant complication of the overall exposition.

2.2.1. *Propositional languages and logical matrices.* Let Σ be a *propositional* (viz., *sentential* or functional) *signature*, that is, a set of function symbols of finite arity to be viewed as *propositional* (viz., *sentential*) *connectives*, and $\mathfrak{Fm}_\Sigma$ the absolutely-free Σ-algebra, freely generated by the countable set of *propositional* (viz., *sentential*) *variables* $V_\omega \triangleq \{p_k\}_{k\in\omega}$. Then, we have the *propositional* (viz., *sentential*) *language* $P_\Sigma \triangleq \langle Fm_\Sigma, \mathrm{hom}(\mathfrak{Fm}_\Sigma, \mathfrak{Fm}_\Sigma)\rangle$ *over* Σ *(and* V_ω*).* (When dealing with indices, we normally write Σ for P_Σ.) A *(logical)* Σ*-matrix* (cf., e.g., [16]) is any couple of the form $\mathcal{A} = \langle\mathfrak{A}, D^{\mathcal{A}}\rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying* one *of* $\mathcal{A}$, and $D^{\mathcal{A}} \subseteq A$. In general, matrices are denoted by Calligraphic letters (possibly, with indices), their underlying algebras being denoted by corresponding Fraktur letters (with same indicies, if any).[3]

2.3. **Disjunctive calculi.** Fix any binary operation δ on Fm_L. Given any $\Gamma, \Delta \subseteq \mathrm{Fm}_L$, put $\delta(\Gamma, \Delta) \triangleq \delta[\Gamma \times \Delta]$.

An L-calculus $\mathbb{C}$ is said to be δ*-disjunctive*, provided

$$(\mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\}) \cap \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\Psi\})) = \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\delta(\Phi, \Psi)\}), \tag{3}$$

for all $\Gamma \subseteq \mathrm{Fm}_L$ and all $\Phi, \Psi \in \mathrm{Fm}_L$. Further, an L-valuation v is said to be *(strongly)/weakly* δ*-disjunctive*, provided

$$((\{\Phi, \Psi\} \cap v) \neq \varnothing) \iff / \Rightarrow (\delta(\Phi, \Psi) \in v), \tag{4}$$

for all $\Phi, \Psi \in \mathrm{Fm}_L$.

Proposition 2.5. *Let* $\mathbb{C}$ *be an* L*-calculus. Then, the following are equivalent:*

(i) $\mathbb{C}$ *is* δ*-disjunctive;*

(ii) *for all* $\Gamma \subseteq \mathrm{Fm}_L$*, and all* $\Phi, \Psi \in \mathrm{Fm}_L$*, it holds that:*

$$\delta(\Phi, \Psi) \in \mathrm{Cn}_{\mathbb{C}}(\{\Phi\}), \tag{5}$$

$$\delta(\Phi, \Psi) \in \mathrm{Cn}_{\mathbb{C}}(\{\Psi\}), \tag{6}$$

$$\delta(\mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\}), \Psi) \subseteq \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\delta(\Phi, \Psi)\}), \tag{7}$$

$$\delta(\Psi, \Phi) \subseteq \mathrm{Cn}_{\mathbb{C}}(\{\delta(\Phi, \Psi)\}), \tag{8}$$

$$\Phi \in \mathrm{Cn}_{\mathbb{C}}(\{\delta(\Phi, \Phi)\}); \tag{9}$$

(iii) *either* (5) *or* (6) *as well as each of* (7)*,* (8) *and* (9) *hold.*

Proof. First, assume (i) holds. Then, by (3) with $\Gamma = \varnothing$(and $\Psi = \Phi$), we get (5), (6), (8)(and (9)). Further, consider any $\Upsilon \in \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\})$, in which case, by (5), we have $\delta(\Upsilon, \Psi) \in \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\})$. Moreover, by (6), we also have $\delta(\Upsilon, \Psi) \in \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\Psi\})$. Hence, by (3), we eventually get $\delta(\Upsilon, \Psi) \in \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\delta(\Phi, \Psi)\})$, so (7) holds, and so does (ii).

Next, (ii)⇒(iii) is trivial. Conversely, (iii)⇒(ii) is by the equivalence of (5) and (6) under (8).

[3]This convention equally concerns [many-place] {fuzzy} matrices to be defined below.

Finally, assume (ii) holds. Then, (5) and (6) yield the inclusion from from right to left in (3). Conversely, consider any $\Upsilon \in (\mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\}) \cap \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\Psi\}))$. Then, by (7), (8) and (9), we have, respectively, both $\delta(\Psi, \Upsilon) \in \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\delta(\Phi, \Psi)\})$ and $\Upsilon \in \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\delta(\Psi, \Upsilon)\})$, in which case we get $\Upsilon \in \mathrm{Cn}_{\mathbb{C}}(\Gamma \cup \{\delta(\Phi, \Psi)\})$, so the inclusion from from left to right in (3) holds, and so does (i), as required. ∎

Lemma 2.6. *Let $\mathbb{C}$ be a finitary δ-disjunctive L-calculus. Then, the set of all δ-disjunctive elements of* $\mathrm{Cl}(\mathrm{Cn}_{\mathbb{C}})$ *is a closure basis of* $\mathrm{Cl}(\mathrm{Cn}_{\mathbb{C}})$.

Proof. Consider any $X \in \mathrm{Cl}(\mathrm{Cn}_{\mathbb{C}})$. Let $\mathcal{G}_X$ be the set of all δ-disjunctive elements of $\mathrm{Cl}(\mathrm{Cn}_{\mathbb{C}})$ including X. Then, $X \subseteq \bigcap \mathcal{G}_X$. For proving the converse, consider any $\Upsilon \in (\mathrm{Fm}_L \setminus X)$. Set $\mathcal{H}_X^{\Upsilon} \triangleq \{Z \in \mathrm{Cl}(\mathrm{Cn}_{\mathbb{C}}) | X \subseteq Z \not\ni \Upsilon\}$. Then, as $\mathbb{C}$ is finitary, $\mathrm{Cn}_{\mathbb{C}}$ is inductive, in which case $\mathrm{Cl}(\mathrm{Cn}_{\mathbb{C}})$ is inductive as well, and so is $\mathcal{H}_X^{\Upsilon}$. Moreover, $X \in \mathcal{H}_X^{\Upsilon}$. Hence, by Zorn's Lemma, there is some $W \in \max(\mathcal{H}_X^{\Upsilon})$. Let us prove, by contradiction, that W is δ-disjunctive. For suppose W is not δ-disjunctive. On the other hand, $W \in \mathrm{Cl}(\mathrm{Cn}_{\mathbb{C}})$ is weakly δ-disjunctive, in view of Proposition 2.5(i)⇒(ii)(5),(6). Therefore, there are some $\Phi, \Psi \in (\mathrm{Fm}_L \setminus W)$ such that $\delta(\Phi, \Psi) \in W$. Then, in view of the maximality of W, for every $\Omega \in \{\Phi, \Psi\}$, we have $\mathrm{Cn}_{\mathbb{C}}(W \cup \{\Omega\}) \notin \mathcal{H}_X^{\Upsilon}$, in which case $\Upsilon \in \mathrm{Cn}_{\mathbb{C}}(W \cup \{\Omega\})$, for $X \subseteq W \subseteq \mathrm{Cn}_{\mathbb{C}}(W \cup \{\Omega\}) \in \mathrm{Cl}(\mathrm{Cn}_{\mathbb{C}})$. Hence, by the δ-disjunctivity of $\mathbb{C}$, we get $\Upsilon \in W$, contrary to the fact that $W \in \mathcal{H}_X^{\Upsilon}$. Therefore, W is δ-disjunctive, in which case it belongs to $\mathcal{G}_X$. Thus, $\Upsilon \notin \bigcap \mathcal{G}_X$, as required. ∎

As a consequence of Lemmas 2.1, 2.6 and (1), we then get:

Corollary 2.7. *The consequence of any finitary δ-disjunctive L-calculus $\mathbb{C}$ is defined by the set of all δ-disjunctive elements of* $\mathrm{Val}(\mathbb{C})$.

2.4. **Many-fold L-fuzzy sets.** Fix any $n \in (\omega \setminus 1)$.

A *[complemented/"({completely}) distributive"/crisp] (completely) ⟨L-⟩fuzzy set* is a triple of the form $F = \langle B^F, \mathfrak{L}^F, \mu^F \rangle$, where B^F is a set, said to be the *basic* one *of* F, while $\mathfrak{L}^F$ is a [complemented/"({completely}) distributive"/two-element] bounded (more specifically, complete) lattice, referred to as the *grading* one *of* F, whereas $\mu^F : B^F \to L^F$, called the *membership function of* F [//in which case it is uniquely determined — up to isomorphism of two-element lattices — and, for this reason, naturally identified with $(\mu^F)^{-1}[\{1^{\mathfrak{L}^F}\}] \subseteq B^F$]. Then, given any set A and any $g : A \to B^F$, we have the [complemented/"({completely}) distributive"/crisp] (completely) fuzzy set $g^{-1}[F]$ with the same grading lattice, the basic set A and the membership function $g \circ \mu^F$. Further, given sets A and J as well as a J-tuple $\vec{F}$ of [complemented/"({completely}) distributive"] (completely) fuzzy [({in particular, crisp})] sets with basic set A, its *fuzzy direct product* $\prod_A \vec{F}$ *over* A is defined to be the [complemented/"({completely}) distributive"] (completely) fuzzy set with the same basic set A, the grading lattice $\prod_{j \in J} \mathfrak{L}^{F_j}$ and the membership function $\prod_{j \in J} \mu^{F_j}$.

A *[complemented/"({completely}) distributive"/crisp] n-fold (completely) ⟨L-⟩ fuzzy set* is any n-tuple $\vec{F}$ of [complemented/"({completely}) distributive"/crisp] (completely) fuzzy sets with same basic set and grading lattice, said to be the ones *of* $\vec{F}$, that is uniquely determined and, for this reason, naturally identified with the $(n+2)$-tuple $\langle B^{F_0}, \mathfrak{L}^{F_0}, \mu^{F_i} \rangle_{i \in n}$. This is said to be *$\mathcal{N}$-reflexive|-transitive*, where $\mathcal{N} \subseteq (\wp(n) \setminus 1)$, provided, for each $a \in B^{F_0}$ and all $N \in \mathcal{N}$, it holds that $(\bigvee | \bigwedge)_{i \in N}^{\mathfrak{L}^{F_0}} \mu^{F_i}(a) = (1|0)^{\mathfrak{L}^{F_0}}$. Then, given any sets A and J as well as a J-tuple $\vec{\vec{F}}$ of ⟨$\mathcal{N}$-reflexive|transitive⟩ [complemented/"({completely}) distributive"] n-fold (completely) fuzzy sets with basic set A, we have its *fuzzy direct product over* A defined point-wise $\prod^A \vec{\vec{F}} \triangleq \langle \prod^A \vec{F}_i \rangle_{i \in n}$ and being an ⟨$\mathcal{N}$-reflexive|-transitive⟩

[complemented/"({completely}) distributive"] n-fold (completely) fuzzy set with basic set A.

It is remarkable that a 2-fold crisp fuzzy set $\vec{F}$ is $\{2\}$-reflexive|-transitive iff $(B^{F_0} \setminus F_0) \subseteq | \supseteq F_1$. This justifies the following notion.

A 2-fold (crisp) fuzzy set $\vec{F}$ is said to be *left-dual reflexive/transitive*, provided, for each $a \in B^{F_0}$, it holds that $\mu^{F_0}(a)(\leqslant / \geqslant)^{\mathfrak{L}^{F_0}} \mu^{F_1}(a)$ (that is, $\langle B^{F_0} \setminus F_0, F_1 \rangle$ is $\{2\}$-reflexive/transitive). {Note that both left-dual reflexive and transitive 2-fold fuzzy sets are actually ordinary — 1-fold — ones.} This well fits [6, 7], when treating $F_{0/1}$ as necessity measure, while treating $F_{1/0}$ as possibility one.

3. Advanced applications to sequent calculi

Fix any formal language L and any $n \in (\omega \setminus 1)$. Elements of $\mathrm{Seq}_L^n \triangleq \wp_\omega(\mathrm{Fm}_L)^n$ are referred to as *n-place L-sequents*. Any $\sigma \in \mathrm{Sb}_L$ determines the equally-denoted unary operation on Seq_L^n by setting $\sigma(\vec{\Phi}) \triangleq \langle \sigma[\Phi_i] \rangle_{i \in n}$, for all $\vec{\Phi} \in \mathrm{Seq}_L^n$. In this way, we get the formal language $S_L^n \triangleq \langle \mathrm{Seq}_L^n, \mathrm{Sb}_L \rangle$, called the *$n$-place sequentialization of L*. Then, S_L^n-rules/-axioms/-calculi/-valuations/-semantics/-consequences are reffered to as *n-place L-sequent rules/axioms/calculi/valuations/semantics/consequences*.

Given any $N \subseteq n$, any $\Gamma \in \wp_\omega(\mathrm{Fm}_L)$ and any $\vec{\Phi}, \vec{\Psi} \in \mathrm{Seq}_L^n$, put $[N/\Gamma] \triangleq ((N \times \{\Gamma\}) \times ((n \setminus N) \times 1)) \in \mathrm{Seq}_L^n \ni \emptyset \triangleq [n/\varnothing]$ and define $(\vec{\Phi} \uplus \vec{\Psi}) \in \mathrm{Seq}_L^n$ component-wise, as follows: $\pi_i(\vec{\Phi} \uplus \vec{\Psi}) \triangleq (\Phi_i \cup \Psi_i)$, for all $i \in n$. We adopt the conventions of Subsection 2.3 as for δ with regard to the binary operation $\uplus$ on Seq_L^n.

An n-place L-sequent calculus $\mathbb{G}$ is said to be *multiplicative*, provided, for every S_L^n-instance $\Gamma \to \Phi$ of any rule of $\mathbb{G}$ and all $\Psi \in \mathrm{Seq}_L^n$, it holds that $(\Gamma \uplus \Psi) \vdash_{\mathbb{G}} (\Phi \uplus \Psi)$, in which case, when $\mathbb{G}$ is finitary, for any $\mathbb{G}$-derivation ∂ from any $\Delta \subseteq \mathrm{Seq}_L^n$, $\partial \circ (\uplus \Psi)$ is a $(\vdash_{\mathbb{G}} \restriction \omega)$-derivation from $\Delta \uplus \Psi$, and so, by Proposition 2.4, we get:

$$(\mathrm{Cn}_{\mathbb{G}}(\Delta) \uplus \Psi) \subseteq \mathrm{Cn}_{\mathbb{G}}(\Delta \uplus \Psi). \tag{10}$$

Non-trivial n-place L-sequent rules of the form $\Phi \to (\Psi \uplus \Phi)$, where $\Phi, \in \mathrm{Seq}_L \ni \Psi \neq \emptyset$, are referred to as *"basic structural"/"Enlargement instances"*, the set of all them being denoted by $\mathbb{B}_L^n$.

Let $\mathcal{N} \subseteq (\wp(n) \setminus 1)$. Then, n-place L-sequent axioms/"proper rules" of the form $([N/\varphi] \uplus \Phi)/(\{[i/\varphi] \uplus \Phi | i \in N\} \to \Phi)$, where $N \in \mathcal{N}$, $\varphi \in \mathrm{Fm}_L$ and $\Phi \in \mathrm{Seq}_L^n$, are called *$\mathcal{N}$-Sharings/-Cuts*, the set of all them being denoted by $(\mathbb{S}/\mathbb{C})_L^n(\mathcal{N})$.[4]

Proposition 3.1. *Let $\mathcal{N}, \mathcal{M} \in \wp(\wp(n) \setminus 1)$. Then, $[\mathbb{B}_L^n \cup] \mathbb{S}_L^n(\mathcal{N}) \cup \mathbb{C}_L^n(\mathcal{M})$ is multiplicative.*

Proof. Let $\mathbb{G}$ be either $[\mathbb{B}_L^n$ or$]$ $\mathbb{S}_L^n(\mathcal{N})$ or $\mathbb{C}_L^n(\mathcal{M})$. Then, $\mathbb{G}$ is schematic. Moreover, for every $(\Gamma \to \Phi) \in \mathbb{G}$ and each $\Psi \in \mathrm{Seq}_L^n$, $((\Gamma \uplus \Psi) \to (\Phi \uplus \Psi)) \in \mathbb{G}$. Therefore, $\mathbb{G}$ is multiplicative, as required. ∎

An $\lfloor\mathcal{N}$-*reflexive*$|$-*transitive*, where $\mathcal{N} \subseteq (\wp(n) \setminus 1)\rfloor$ *n-place (complemented) [$\{\langle$completely$\rangle\}$ distributive] "$\{$completely$\}$ fuzzy"/crisp L-valuation* is any n-fold $\lfloor\mathcal{N}$-reflexive$|$-transitive$\rfloor$ (complemented) [$\{\langle$completely$\rangle\}$ distributive] "$\{$completely$\}$ fuzzy"/crisp set $\vec{F}$ with basic set Fm_L. This determines the n-place L-sequent valuation $(\vec{F}\dagger) \triangleq \{\vec{\Phi} \in \mathrm{Seq}_L^n \mid \bigvee^{\mathfrak{L}^{F_0}}(\bigcup_{i \in n} \mu^{F_i}[\Phi_i]) = 1^{\mathfrak{L}^{F_0}}\}$. Any n-place L-sequent $\lceil$semantics consisting of n-place L-sequent$\rceil$ valuation$\lceil$s$\rceil$ of such a kind is referred to as $\lfloor\mathcal{N}$-*reflexive*$|$-*transitive*$\rfloor$ *(complemented-)[$\{\langle$completely-$\rangle\}$distributive-]"$\{$completely-$\}$fuzzy-"/crisp-decomposable*.

[4]This covers all miscellaneous systems of many-place sharings and cuts (cf. [52], [53], [32], [39]).

Remark 3.2. It is routine checking that the consequence of any fuzzy-/crisp-decomposable n-place L-sequent semantics /"is multiplicative and" contains all basic structural rules. ■

Lemma 3.3. *Any finitary multiplicative n-place L-sequent calculus $\mathbb{G}$ with derivable basic structural rules is $\uplus$-disjunctive.*

Proof. With using Propositions 2.4 and 2.5(iii)⇒(i). First, (6) is by basic structural rules. Next, (8) and (9) are by the fact that $(\Phi \uplus \Psi) = (\Psi \uplus \Phi)$ and $(\Phi \uplus \Phi) = \Phi$, for all $\Phi, \Psi \in \mathrm{Seq}^n_L$. Finally, consider any $((\Gamma \cup \{\Phi\}) \to \Upsilon) \in \vdash_{\mathbb{G}}$ and any $\Psi \in \mathrm{Seq}^n_L$. Then, by basic structural rules and (10), we have $(\Gamma \cup \{\Phi \uplus \Psi)\}) \vdash_{\mathbb{G}} ((\Gamma \cup \{\Phi\}) \uplus \Psi) \vdash_{\mathbb{G}} (\Upsilon \uplus \Psi)$, so (7) holds, as required. ■

Lemma 3.4. *Any proper $\uplus$-disjunctive n-place L-sequent valuation v is crisp-decomposable.*

Proof. Define the crisp n-place L-valuation $\vec{c}$ as follows: for every $i \in n$, put $c_i \triangleq \{\varphi \in \mathrm{Fm}_L \mid [i/\varphi] \in v\}$. Consider any $\vec{\Phi} \in \mathrm{Seq}^n_L$. First, assume $\vec{\Phi} \in (\vec{c}\dagger)$. Then, there are some $i \in n$ and some $\varphi \in (\Phi_i \cap c_i)$, in which case we have $[i/\varphi] \in v$, and so by the $\uplus$-disjunctivity of v, we get $\vec{\Phi} = (\vec{\Phi} \uplus [i/\varphi]) \in v$. Conversely, assume $\vec{\Phi} \notin (\vec{c}\dagger)$. Then, for every $i \in n$ and each $\varphi \in \Phi_i$, it holds that $\varphi \notin c_i$, that is $[i/\varphi] \notin v$. Moreover, $\emptyset \notin v$, for, otherwise, in view of the $\uplus$-disjunctivity of v, for every $\Psi \in \mathrm{Seq}^n_L$, we would have $\Psi = (\Psi \uplus \emptyset) \in v$, and so v would be total. Hence, for each $i \in n$, we have some bijection $\bar{\phi}$ from $m_i \triangleq |\Phi_i| \in \omega$ onto Φ_i, in which case, by the $\uplus$-disjunctivity of v, we have $[i/\Phi_i] = \uplus^{\emptyset}\langle [i/\phi_j]\rangle_{j \in m_i} \notin v$, and so we get $\vec{\Phi} = \uplus\langle [i/\Phi_i]\rangle_{i \in n} \notin v$. Thus, $v = (\vec{c}\dagger)$, as required. ■

Lemma 3.5. *Let $\vec{F}$ be an n-place fuzzy L-valuation and $\mathcal{N} \subseteq (\wp(n) \setminus 1)$. [Suppose $\vec{F}\dagger$ is proper.] Then, $(\vec{F}\dagger) \in \mathrm{Val}((\mathbb{S}|\mathbb{C})^n_L(\mathcal{N}))$ if[f] $\vec{F}$ is $\mathcal{N}$-reflexive|-transitive.*

Proof. The "if" part is immediate. [The converse is so as well, in view of the fact that $\emptyset \notin (\vec{F}\dagger)$, taking Remark 3.2 into account.] ■

Now, we are in a position to prove the key result of the paper:

Corollary 3.6. *Let $\mathbb{G}$ be an n-place L-sequent calculus and $\mathcal{N}, \mathcal{M} \in \wp(\wp(n) \setminus 1)$. [Suppose $\mathbb{G}$ is finitary.] Then, $\mathbb{G}$ is multiplicative and $(\mathbb{B}^n_L \cup \mathbb{S}^n_L(\mathcal{N}) \cup \mathbb{C}^n_L(\mathcal{M})) \subseteq \vdash_{\mathbb{G}}$ if[f] $\vdash_{\mathbb{G}}$ is defined by an $\mathcal{N}$-reflexive $\mathcal{M}$-transitive crisp-decomposable n-place L-sequent semantics.*

Proof. The "if" part is by Remark 3.2 and Lemma 3.5. [The converse is by Lemmas 3.3, 3.4, 3.5, Corollary 2.7, Proposition 2.4 and (2).] ■

As an immediate consequence of Corollary 3.6 and Proposition 3.1, we first get:

Corollary 3.7. *Let $\mathcal{N}, \mathcal{M} \in \wp(\wp(n) \setminus 1)$. Then, the consequence of $\mathbb{B}^n_L \cup \mathbb{S}^n_L(\mathcal{N}) \cup \mathbb{C}^n_L(\mathcal{M})$ is defined by the set of all $\mathcal{N}$-reflexive $\mathcal{M}$-transitive crisp-decomposable n-place L-sequent valuations.*

Given any $\sigma \in \mathrm{Sb}_L$ and [any $\mathcal{N} \subseteq (\wp(n) \setminus 1)$ as well as] any (complemented/"{⟨completely⟩} distributive"/crisp) [$\mathcal{N}$-reflexive|-transitive] n-place {completely} fuzzy L-valuation $\vec{F}$, we have one $\sigma^{-1}[\vec{F}] \triangleq \langle \sigma^{-1}[F_i]\rangle_{i \in n}$ such that:

$$(\sigma^{-1}[\vec{F}]\dagger) = \sigma^{-1}[\vec{F}\dagger]. \tag{11}$$

Likewise, given any set J and any J-tuple $\overline{\vec{F}^j}$ of (complemented/"{⟨completely⟩} distributive") [$\mathcal{N}$-reflexive|-transitive] n-place {completely} fuzzy L-valuations, we have one $\prod^{\mathrm{Fm}_L} \overline{\vec{F}}$ such that:

$$((\prod\nolimits^{\mathrm{Fm}_L} \overline{\vec{F}})\dagger) = (\mathrm{Seq}^n_L \cap \bigcap\nolimits_{j \in J} (\vec{F}^j\dagger)). \tag{12}$$

In this way, combining Corollary 3.7, Theorem 2.3, Lemma 2.1, (11) and (12), we get:

Corollary 3.8. *Let* $\mathcal{N},\mathcal{M} \in \wp(\wp(n)\setminus 1)$. *Then, any element of* $\mathrm{Val}(\mathbb{B}^n_L \cup \mathbb{S}^n_L(\mathcal{N}) \cup \mathbb{C}^n_L(\mathcal{M}))$ *is* $\mathcal{N}$*-reflexive* $\mathcal{M}$*-transitive [complemented-]({completely-})distributive-(completely-)fuzzy-decomposable. In particular, any extension of* $\mathbb{B}^n_L \cup \mathbb{S}^n_L(\mathcal{N}) \cup \mathbb{C}^n_L(\mathcal{M})$ *is defined by an* $\mathcal{N}$*-reflexive* $\mathcal{M}$*-transitive [complemented-]({completely-})distributive-(completely-)fuzzy-decomposable* n*-place* L*-sequent semantics.*

After all, Remark 3.2, Lemma 3.5 and Corollaries 3.6 and 3.8 yield the main generic result of the paper:

Theorem 3.9. *Let* $\mathbb{G}$ *be an* n*-place* L*-sequent calculus and* $\mathcal{N},\mathcal{M} \in \wp(\wp(n)\setminus 1)$. *[Suppose* $\mathbb{G}$ *is finitary.] Then, /"*$\mathbb{G}$ *is multiplicative and"* $\mathbb{B}^n_L \cup \mathbb{S}^n_L(\mathcal{N}) \cup \mathbb{C}^n_L(\mathcal{M}) \subseteq \vdash_{\mathbb{G}}$ *iff/if[f]* $\vdash_{\mathbb{G}}$ *is defined by an* $\mathcal{N}$*-reflexive* $\mathcal{M}$*-transitive "(complemented-){⟨completely-⟩}distributive-{completely-}fuzzy-"/crisp-decomposable* n*-place* L*-sequent semantics.*

3.1. **The propositional case and many-place fuzzy matrices.**

Fix any propositional signature Σ. We write "propositional Σ-sequent" for "P_Σ-sequent".

An {$\mathcal{N}$*-reflexive*|*-transitive*, where $\mathcal{N} \subseteq (\wp(n)\setminus 1)$} *$n$-place (complemented) [⟨⌈completely-⌉⟩distributive] "⟨completely-⟩fuzzy"/crisp Σ-matrix* is any couple of the form $\mathcal{A} = \langle \mathfrak{A}, \vec{F}^{\mathcal{A}} \rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of* $\mathcal{A}$, and $\vec{F}^{\mathcal{A}}$ is an{ $\mathcal{N}$-reflexive/-transitive} n-fold (complemented) [⟨⌈completely-⌉⟩distributive] "⟨completely-⟩fuzzy"/crisp set with basic set A, called the one *of* $\mathcal{A}$.[5] Then, given any subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, put $(\mathcal{A} \upharpoonright B) = \langle \mathfrak{B}, \langle F^{\mathcal{A}}_i \upharpoonright B \rangle_{i \in n} \rangle$.

Given any class M of {$\mathcal{N}$-reflexive|-transitive} n-place (complemented) [⟨⌈completely-⌉⟩distributive] "⟨completely-⟩fuzzy"/crisp Σ-matrices, we have the {$\mathcal{N}$-reflexive|-transitive} [⟨⌈completely-⌉⟩distributive-]"⟨completely-⟩fuzzy-"/crisp-decomposable n-place propositional Σ-semantics $\mathsf{S}_\dagger(\mathsf{M}) \triangleq \{h^{-1}[\vec{F}^{\mathcal{A}}] \dagger \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{A})\}$, in which case $\vdash_{\mathsf{M}} \triangleq \vdash_{\mathsf{S}_\dagger(\mathsf{M})}$ is said to be *defined by* M.

Conversely, given a set S of {$\mathcal{N}$-reflexive|-transitive} n-place (complemented) [⟨⌈completely-⌉⟩distributive] "⟨completely-⟩fuzzy"/crisp P_Σ-valuations, we have the class $\mathsf{M}(\mathsf{S}) \triangleq \{\langle \mathfrak{Fm}_\Sigma, \vec{F} \rangle \mid \vec{F} \in \mathsf{S}\}$ of {$\mathcal{N}$-reflexive|-transitive} n-place [⟨⌈completely-⌉⟩distributive] "⟨completely-⟩fuzzy"/crisp Σ-matrices, in which case $\mathsf{S}_\dagger(\mathsf{M}(\mathsf{S})) = \mathbf{S}^{-1}([\mathsf{S}]\dagger)$, in view of (11), and so, by Lemma 2.2, we get:

$$\vdash_{\mathsf{M}(\mathsf{S})} = \vdash_{[\mathsf{S}]\dagger}, \tag{13}$$

as $\mathbf{S}^{-1}$ is idempotent, because $\Delta_{\mathrm{Fm}_L} \in \mathrm{Sb}_L$, for any formal language L. In this way, Theorem 3.9 and (13) immediately yield:

Corollary 3.10. *Let* $\mathbb{G}$ *be an* n*-place propositional* Σ*-sequent calculus and* $\mathcal{N},\mathcal{M} \in \wp(\wp(n)\setminus 1)$. *[Suppose* $\mathbb{G}$ *is finitary.] Then, /"*$\mathbb{G}$ *is multiplicative and"* $(\mathbb{B}^n_\Sigma \cup \mathbb{S}^n_\Sigma(\mathcal{N}) \cup \mathbb{C}^n_\Sigma(\mathcal{M})) \subseteq \vdash_{\mathbb{G}}$ *iff/if[f]* $\vdash_{\mathbb{G}}$ *is defined by a class of* $\mathcal{N}$*-reflexive* $\mathcal{M}$*-transitive* n*-place "{complemented} (⟨completely-⟩)distributive (completely-)fuzzy"/crisp* Σ*-matrices.*

The []-optional "crisp" case of this corollary has been actually due to [32] and [39].

3.2. **Peculiarities of the two-side case.**

Here, it is supposed that $n = 2$. In that case, "(left/right side)" means "place (0/1)". Any two-side sequent $\vec{\Phi}$ is written in the conventional form $\Phi_0 \rightarrowtail \Phi_1$ involving the binary infix "side-separator" symbol $\rightarrowtail$ (instead of the traditional ones $\vdash$ and $\rightarrow$ just to avoid a confusion with equally traditional notations of consequence relations and rules, respectively).

[5] The crisp case corresponds to [32] and [39].

The basic peculiarity of the two-side case is that, as opposed to the general many-place one dealing with rather miscellaneous systems of Sharings and Cuts (cf. [52, 53, 32, 39]), two-side ones are unambiguous and are determined by the sets $\mathcal{N} = \mathcal{M} = \{2\}$, which are normally no longer mentioned explicitly from now on. This justifies an alternative way of associating 2-side L-sequent valuations with 2-side fuzzy L-valuations that fits both [27] and [30].

Any $\langle$left-dual reflexive$|$-transitive$\rangle$ 2-side (complemented) $[\{\lceil$completely-$\rceil\}$distributive$]$ "$\{$completely$\}$ fuzzy"/crisp L-valuation determines the 2-side L-sequent valuation $(\vec{F}\ddagger) \triangleq \{\vec{\Phi} \in \mathrm{Seq}^2_L \mid \bigwedge^{\mathfrak{L}^{F_0}}(\mu^{F_0}[\Phi_0]) \leqslant^{\mathfrak{L}^{F_0}} \bigvee^{\mathfrak{L}^{F_0}}(\mu^{F_1}[\Phi_1])\}$ /"in which case:

$$(\vec{F}\ddagger) = (\langle \complement_L(F_0), F_1\rangle\dagger)\text{"} \tag{14}$$

Then, any 2-side L-sequent $\lfloor$semantics consisting of 2-side L-sequent$\rfloor$ valuation$\lfloor$s$\rfloor$ of such a kind is referred to as *left-dual $\langle$reflexive$|$-transitive$\rangle$ "(complemented-)$[\{\lceil$completely-$\rceil\}$distributive-$\{$completely-$\}$fuzzy-"/crisp-decomposable*. In view of (14), we then get the following "left-dual" version of the 2-side case of Corollary 3.6:

Corollary 3.11. *Let $\mathbb{G}$ be a [finitary] 2-side L-sequent calculus. Then, $\mathbb{G}$ is multiplicative and $(\mathbb{B}^2_L\{\cup\mathbb{S}^2_L\}(\cup\mathbb{C}^2_L)) \subseteq \vdash_{\mathbb{G}}$ if[f] $\vdash_{\mathbb{G}}$ is defined by a left-dual $\{$reflexive$\}$ (transitive) crisp-decomposable 2-side L-sequent semantics.*

As an immediate consequence of Corollary 3.11 and Proposition 3.1, we then get:

Corollary 3.12. *The consequence of $\mathbb{B}^2_L\langle(\cup\mathbb{S}^2_L)[\cup\mathbb{C}^2_L]$ is defined by the set of all left-dual (reflexive) [transitive] crisp-decomposable 2-side L-sequent valuations.*

Given any $\sigma \in \mathrm{Sb}_L$ and any [left dual reflexive$|$-transitive] 2-side "($\{$completely$\}$) distributive (completely) fuzzy"/crisp L-valuation $\vec{F}$, $\sigma^{-1}[\vec{F}]$ is one such that:

$$(\sigma^{-1}[\vec{F}]\ddagger) = \sigma^{-1}[\vec{F}\ddagger]. \tag{15}$$

Likewise, given any set J and any J-tuple $\overline{\vec{F}}$ of [left dual reflexive$|$-transitive] 2-side ($\{$completely$\}$) distributive (completely) fuzzy L-valuations, $\prod^{\mathrm{Fm}_L}\overline{\vec{F}}$ is one such that:

$$((\prod^{\mathrm{Fm}_L}\overline{\vec{F}})\ddagger) = (\mathrm{Seq}^2_L \cap \bigcap\nolimits_{j\in J}(\vec{F}_j\ddagger)). \tag{16}$$

In this way, combining Corollary 3.12, Theorem 2.3, Lemma 2.1, (15) and (16), we get:

Corollary 3.13. *Any element of $\mathrm{Val}(\mathbb{B}^2_L(\cup\mathbb{S}^2_L)[\cup\mathbb{C}^2_L])$ is left-dual (reflexive) [transitive] $\lfloor$complemented$\rfloor$ $\{\langle$completely-$\rangle\}$distributive-$\{$completely-$\}$fuzzy-decomposable. In particular, any extension of $\mathbb{B}^2_L(\cup\mathbb{S}^n_L)[\cup\mathbb{C}^2_L]$ is defined by a left-dual (reflexive) [transitive] $\lfloor$complemented-$\rfloor\{\langle$completely-$\rangle\}$distributive-$\{$completely-$\}$fuzzy-decomposable 2-side L-sequent semantics.*

After all, Remark 3.2, Lemma 3.5 and Corollaries 3.11 and 3.13 yield the main generic result of the paper concerining two-side sequent calculi:

Theorem 3.14. *Let $\mathbb{G}$ be a $\{$finitary$\}$ 2-side L-sequent calculus. Then, /"$\mathbb{G}$ is multiplicative and" $(\mathbb{B}^2_L(\cup\mathbb{S}^2_L)[\cup\mathbb{C}^2_L]) \subseteq \vdash_{\mathbb{G}}$ iff/if$\{f\}$ $\vdash_{\mathbb{G}}$ is defined by a left-dual (reflexive) [transitive] "$\lfloor$complemented-$\rfloor\langle\lceil$completely-$\rceil\rangle$distributive-$\langle$completely-$\rangle$fuzzy-"/ crisp-decomposable 2-side L-sequent semantics.*

The optional case of this theorem (dealing with both Sharings and Cuts) has been due to [27]. However, the argumentation found therein is not applicable to proving the theorem under consideration in the general case, because it was essentially based upon presence of both Sharings and Cuts. This is why the theorem involved is a substantial advance of the present study with regard to [27]. This remark equally concerns Corollary 3.15 below.

3.2.1. *The propositional case.* Here, we properly follow Subsection 3.1.

A 2-side fuzzy Σ-matrix $\mathcal{A} = \langle \mathfrak{A}, \vec{F}^{\mathcal{A}} \rangle$ is said to be *left-dual reflexive/-transitive*, whenever $\vec{F}^{\mathcal{A}}$ is so.

Given any class M of {left-dual reflexive|transitive} 2-side⌊complemented⌋ [⌈⟨completely⌉⟩ distributive] "⟨completely⟩ fuzzy"/crisp Σ-matrices, we have the left-dual {reflexive|-transitive} ⌊complemented⌋ [⟨⌈completely⌉⟩-distributive-]"⟨completely-⟩fuzzy-"/crisp-)decomposable 2-side propositional Σ-semantics $\mathsf{S}_{\ddagger}(\mathsf{M}) \triangleq \{h^{-1}[\vec{F}^{\mathcal{A}}] \ddagger \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}_{\Sigma}, \mathfrak{A})\}$, in which case $\vdash^{\mathrm{ld}}_{\mathsf{M}} \triangleq \vdash_{\mathsf{S}_{\ddagger}(\mathsf{M})}$ is said to be *left-dual defined by* M, while, for any set S of {left-dual reflexive|-transitive} 2-side ⌊complemented⌋ [⟨⌈completely⌉⟩ distributive] "⟨completely⟩ fuzzy"/crisp P_{Σ}-valuations, every member of $\mathsf{M}(\mathsf{S})$ is a {left-dual reflexive|transitive} 2-side ⌊complemented⌋ [⟨⌈completely⌉⟩ distributive] "⟨completely⟩ fuzzy"/crisp Σ-matrix, whereas $\mathsf{S}_{\ddagger}(\mathsf{M}(\mathsf{S})) = \mathbf{S}^{-1}([\mathsf{S}]\ddagger)$, in view of (15), and so, by Lemma 2.2, we get:

$$\vdash^{\mathrm{ld}}_{\mathsf{M}(\mathsf{S})} = \vdash_{\mathsf{S}\ddagger}, \tag{17}$$

as $\mathbf{S}^{-1}$ is idempotent, because $\Delta_{\mathrm{Fm}_L} \in \mathrm{Sb}_L$, for any formal language L. In this way, Theorem 3.14 and (17) immediately yield:

Corollary 3.15. *Let $\mathbb{G}$ be a (finitary) 2-side propositional Σ-sequent calculus. Then, /"$\mathbb{G}$ is multiplicative and" $(\mathbb{B}^2_{\Sigma}\langle\cup\mathbb{S}^2_{\Sigma}\rangle[\cup\mathbb{C}^2_{\Sigma}]) \subseteq \vdash_{\mathbb{G}}$ iff/if(f) $\vdash_{\mathbb{G}}$ is defined by a class of ⟨left-dual reflexive⟩ [left-dual transitive] 2-side "⌊complemented⌋ {⌈completely⌉} distributive {completely} fuzzy"/crisp Σ-matrices.*

4. Examples

Here, we deal with the propositional languages $\Sigma^{[(\neg)]}_{[01]} \triangleq \{\wedge, \vee, \sim [(, \neg), \bot, \top]\}$, where $\wedge$ — conjunction — and $\vee$ — disjunction — are binary, while $\sim$ — negation — [(as well as $\neg$ — classical negation)] is unary [whereas $\bot$ and $\top$ — truth and falsehood constants — are nullary].

By $\mathfrak{B}^{[(\neg)]}_{4[,01}$ we denote the $\Sigma^{[(\neg)]}_{[01]}$-algebra with carrier 2^2 and operations defined as follows: put, for all $\vec{a}, \vec{b} \in 2^2$, $(\bar{a}(\wedge|\vee)^{\mathfrak{B}^{[(\neg)]}_{4[,01}}\bar{b}) \triangleq \langle(\min|\max)(a_i, b_i)\rangle_{i\in 2}$ and $(\sim \{/\neg\})^{\mathfrak{B}^{[(\neg)]}_{4[,01}}\bar{a} \triangleq \langle 1 - a_{(1-i)[(/i)]}\rangle_{i\in 2}$ [as well as $(\bot|\top)^{\mathfrak{B}^{[(\neg)]}_{4[,01}} \triangleq \langle 0|1, 0|1\rangle$].[6] In this connection, we use the following standard abbreviations:

$$\mathsf{t} \triangleq \langle 1,1\rangle \qquad \mathsf{f} \triangleq \langle 0,0\rangle \qquad \mathsf{b} \triangleq \langle 0,1\rangle \qquad \mathsf{n} \triangleq \langle 1,0\rangle$$

4.1. **Sharing- and Cut-free versions of Gentzen's calculus versus First-Degree Entailment.** By $\mathbb{LK}^{(\mathrm{S})\{\mathrm{C}\}}_{[01]}$ we denote the two-side propositional $\Sigma_{[01]}$-sequent calculus constituted by basic structural rules (and Sharings) {as well as Cuts} collectively with both the following rules and inverse to these [unless they

[6] This is nothing but FDE [1, 3] (cf. [21, 24, 49]) [expanded by truth and falsehood constants (as well as classical negation)].

are basic structural ones]:

	Left	Right
$(\wedge)$	$\dfrac{(\Gamma\cup\{\phi,\psi\})\rightarrowtail\Delta}{(\Gamma\cup\{\phi\wedge\psi\})\rightarrowtail\Delta}$	$\dfrac{\{\Gamma\rightarrowtail(\Delta\cup\{\phi\});\Gamma\rightarrowtail(\Delta\cup\{\psi\})\}}{\Gamma\rightarrowtail(\Delta\cup\{\phi\wedge\psi\})}$
$(\vee)$	$\dfrac{\{(\Gamma\cup\{\phi\})\rightarrowtail\Delta;(\Gamma\cup\{\psi\})\rightarrowtail\Delta\}}{(\Gamma\cup\{\phi\vee\psi\})\rightarrowtail\Delta}$	$\dfrac{(\Gamma\rightarrowtail(\Delta\cup\{\phi,\psi\})}{\Gamma\rightarrowtail(\Delta\cup\{\phi\vee\psi\})}$
$(\sim)$	$\dfrac{\Gamma\rightarrowtail(\Delta\cup\{\phi\})}{(\Gamma\cup\{\sim\phi\})\rightarrowtail\Delta}$	$\dfrac{(\Gamma\cup\{\phi\})\rightarrowtail\Delta}{\Gamma\rightarrowtail(\Delta\cup\{\sim\phi\})}$
$[(\bot)$	$\bot\rightarrowtail$	$\dfrac{\rightarrowtail\bot}{\rightarrowtail}$
$(\top)$	$\dfrac{\top\rightarrowtail}{\rightarrowtail}$	$\rightarrowtail\top]$

where $\phi,\psi\in\mathrm{Fm}_\Sigma$ and $\Gamma,\Delta\in\wp_\omega(\mathrm{Fm}_\Sigma)$. Note that $\mathbb{LK}^{\mathrm{SC}}_{[01]}$ is the propositional fragment of Gentzen's calculus [10] supplemented with rules inverse to the above *logical* ones, which are derivable in the original calculus, so they have same derivable rules, though such is the case for the neither Cut- nor Sharing-free versions.

Lemma 4.1. $\mathbb{LK}^{\mathrm{(S)\{C\}}}_{[01]}$ *is multiplicative.*

Proof. Note that $\mathbb{G}\triangleq(\mathbb{LK}^{\mathrm{(S)\{C\}}}_{[01]}\setminus(\mathbb{B}^2_{\Sigma_{[01]}}\cup\mathbb{S}^2_{\Sigma_{[01]}}\cup\mathbb{C}^2_{\Sigma_{[01]}}))$ is schematic. Moreover, for any $(\Gamma\to\Phi)\in\mathbb{G}$ and all $\Psi\in\mathrm{Seq}^2_{\Sigma_{[01]}}$, it holds that $((\Gamma\uplus\Psi)\to(\Phi\uplus\Psi))\in\mathbb{G}$. Hence, $\mathbb{G}$ is multiplicative. Then, Proposition 3.1 completes the argument. ∎

Given any $\bar{c}\in\{\mathsf{b},\mathsf{n}\}^*$, we have the subalgebra $\mathfrak{B}_{4[,01]-\bar{c}}$ with carrier $B_{4-\bar{c}}\triangleq(2^2\setminus(\mathrm{img}\,\bar{c}))$, in which case $\mathfrak{B}_{4[,01]-(\mathsf{b}/\mathsf{n}/\mathsf{bn})}$ corresponds to "[the bounded version of] Kleene's three-valued logic [15]"/"[the bounded version of] Priest's *logic of paradox* [19] (cf. [22, 26])"/"the classical logic". Then, we have the 2-side crisp $\Sigma_{[01]}$-matrix $\mathcal{B}_{4[,01]}\triangleq\langle\mathfrak{B}_{4[,01]},\{\mathsf{t},\mathsf{n}\},\{\mathsf{t},\mathsf{b}\}\rangle$. Put $\mathcal{B}_{4[,01]-\bar{c}}\triangleq(\mathcal{B}_{4[,01]}\upharpoonright B_{4-\bar{c}})$. This is clearly left-dual reflexive/transitive iff $(\mathsf{n}/\mathsf{b})\in(\mathrm{img}\,\bar{c})$.

Theorem 4.2 (Completeness Theorem)**.** *The consequence of* $\mathbb{LK}^{\mathrm{(S)\{C\}}}_{[01]}$ *is left-dual defined by* $\mathcal{B}_{4[,01]-(\mathsf{n})\{\mathsf{b}\}}$.

Proof. Note that both $\{\mathsf{t},\mathsf{n}\}$ and $\{\mathsf{t},\mathsf{b}\}$ are prime filters of $\mathfrak{B}_4\upharpoonright\{\wedge,\vee\}$. Hence, $(\wedge)$ and $(\vee)$ are true in $\mathsf{S}_\ddagger(\mathcal{B}_{4[,01]-(\mathsf{n})\{\mathsf{b}\}})$. Moreover, $\sim^{\mathfrak{B}_4}[\{\mathsf{f},\mathsf{b}\}]=\{\mathsf{t},\mathsf{b}\}$, and so $\sim^{\mathfrak{B}_4}[\{\mathsf{t},\mathsf{b}\}]=\{\mathsf{f},\mathsf{b}\}$. Therefore, $(\sim)$ are true in $\mathsf{S}_\ddagger(\mathcal{B}_{4[,01]-(\mathsf{n})\{\mathsf{b}\}})$. [Finally, $(\bot)$ and $(\top)$ are clearly true in $\mathsf{S}_\ddagger(\mathcal{B}_{4[,01]-(\mathsf{n})\{\mathsf{b}\}})$, for $\rightarrowtail$ is not so]. Thus, by Corollary 3.12, we get $\vdash_{\mathbb{LK}^{\mathrm{(S)\{C\}}}_{[01]}}\subseteq\vdash^{\mathrm{ld}}_{\mathcal{B}_{4[,01]-(\mathsf{n})\{\mathsf{b}\}}}$.

Conversely, by Lemma 4.1 and Corollary 3.11, $\vdash_{\mathbb{LK}^{\mathrm{(S)\{C\}}}_{[01]}}$ is left-dual defined by a left-dual (reflexive) {transitive} crisp-decomposable 2-side propositional $\Sigma_{[01]}$-sequent semantics S. Consider any $v\in\mathsf{S}$. Then, there is some left-dual (reflexive) [transitive] 2-place crisp $P_{\Sigma_{[01]}}$-valuation $\vec{C}$ such that $v=(\vec{C}\ddagger)$. In that case, taking $(\sim)$ into account, we have:

$$\{\varphi\in\mathrm{Fm}_{\Sigma'}\,|(\varphi\rightarrowtail\varnothing)\notin v\}=C_0=\{\varphi\in\mathrm{Fm}_{\Sigma'}\mid\sim\varphi\notin C_1\},\tag{18}$$

$$\{\varphi\in\mathrm{Fm}_{\Sigma'}\,|(\varnothing\rightarrowtail\varphi)\in v\}=C_1=\{\varphi\in\mathrm{Fm}_{\Sigma'}\mid\sim\varphi\notin C_0\},\tag{19}$$

where $\Sigma'=\Sigma_{[01]}$. Then, by the left equalities in (18) and (19) as well as $(\wedge)$ and $(\vee)$ for side $i\in 2$, we also get, respectively:

$$((\phi\wedge\psi)\in C_i)\iff(\{\phi,\psi\}\subseteq C_i),\tag{20}$$

$$((\phi\vee\psi)\in C_i)\iff((\{\phi,\psi\}\cap C_i)\neq\varnothing),\tag{21}$$

for all $\phi, \psi \in \mathrm{Fm}_{\Sigma'}$, where $\Sigma' = \Sigma_{[01]}$. [Likewise, as $\rightarrowtail \notin v$, by the left equalities in (18) and (19) as well as both ($\bot$) and ($\top$) for side $i \in 2$, we equally get, respectively:

$$\bot \notin C_i, \tag{22}$$

$$\top \in C_i]. \tag{23}$$

Therefore, by the right equalities in (18) and (19) as well as both (20) and (21) [collectively with both (22) and (23)], taking the following immediate observation into account, we conclude that $h \triangleq (\prod_{i\in 2} \chi^{C_i}_{\mathrm{Fm}_{\Sigma_{[01]}}}) \in \hom(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{B}_{4[,01]})$, while $\vec{C} = h^{-1}[\langle\{\mathsf{t},\mathsf{n}\},\{\mathsf{t},\mathsf{b}\}\rangle]$:

Claim 4.3. *Let $\Sigma' \supseteq \Sigma_{[01]}$ be a propositional language and $\vec{C} \in \wp(\mathrm{Fm}_{\Sigma'})^2$. Suppose the right equalities in* (18) *and* (19) *as well as both* (20) *and* (21) *[collectively with both* (22) *and* (23)*] hold. Then, $h \triangleq (\prod_{i\in 2} \chi^{C_i}_{\mathrm{Fm}_{\Sigma'}}) \in \hom(\mathfrak{Fm}_{\Sigma'} \upharpoonright \Sigma_{[01]}, \mathfrak{B}_{4[,01]})$, while $\vec{C} = h^{-1}[\langle\{\mathsf{t},\mathsf{n}\},\{\mathsf{t},\mathsf{b}\}\rangle]$.*

And what is more, since $C_0(\subseteq)\{\supseteq\}C_1$, we also have $(\mathsf{n})\{\mathsf{b}\} \notin h[\mathrm{Fm}_{\Sigma_{[01]}}]$. In this way, $v \in \mathsf{S}_{\ddagger}(\mathcal{B}_{4[,01]-(\mathsf{n})\{\mathsf{b}\}})$. Thus, $\mathsf{S} \subseteq \mathsf{S}_{\ddagger}(\mathcal{B}_{4[,01]-(\mathsf{n})\{\mathsf{b}\}})$, and so $\vdash_{\mathbb{LK}^{(\mathrm{S})\{\mathrm{C}\}}_{[01]}} \supseteq \vdash^{\mathrm{ld}}_{\mathcal{B}_{4[,01]-(\mathsf{n})\{\mathsf{b}\}}}$, as required. ∎

This strengthens [38] and, in general, demonstrates the power of generic results obtained in the paper.

Corollary 4.4. *There are Cuts/Sharings not derivable in $\mathbb{LK}^{\mathrm{S/C}}_{[01]}$, and so in $\mathbb{LK}_{[01]}$.*

Proof. Let $h \in \hom(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{B}_{4[,01]-(\mathsf{n}/\mathsf{b})})$ extend $V_\omega \times \{\mathsf{b}/\mathsf{n}\}$. Then, $(\{\varnothing \rightarrowtail p_0, p_0 \rightarrowtail \varnothing\} \to \emptyset)/(p_0 \rightarrowtail p_0)$ is not true in $h^{-1}[\langle\{\mathsf{t}\}/\{\mathsf{t},\mathsf{n}\},\{\mathsf{t},\mathsf{b}\}/\{\mathsf{t}\}\rangle]\ddagger$ under $\iota_{S^2_{\Sigma_{[01]}}}$. In this way, Theorem 4.2 completes the argument. ∎

On the other hand, $\mathbb{LK}^{\mathrm{SC}}_{[01]}$ and $\mathbb{LK}^{\mathrm{S}}_{[01]}$ are well-known to have same derivable axioms, and so admissible rules.[7] This can equally be shown with using Theorem 4.2 as follows:

Corollary 4.5 (Cut Elimination Theorem)**.** $\mathrm{Cn}_{\mathbb{LK}^{\mathrm{SC}}_{[01]}}(\varnothing) = \mathrm{Cn}_{\mathbb{LK}^{\mathrm{S}}_{[01]}}(\varnothing)$.

Proof. The inclusion from right to left is trivial, for $\mathbb{LK}^{\mathrm{S}}_{[01]} \subseteq \mathbb{LK}^{\mathrm{SC}}_{[01]}$. Conversely, consider any $(\Gamma \rightarrowtail \Delta) \in \mathrm{Cn}_{\mathbb{LK}^{\mathrm{SC}}_{[01]}}(\varnothing)$. Then, by Theorem 4.2, for each $h \in \hom(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{B}_{4[,01]-\mathsf{nb}})$, either $(h[\Gamma] \cap \{\mathsf{f}\}) \neq \varnothing$ or $(h[\Delta] \cap \{\mathsf{t}\}) \neq \varnothing$. Consider any $g \in \hom(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{B}_{4[,01]-\mathsf{n}})$. Let $\mathfrak{K}_{4[,01}$ be the $\Sigma_{[01]}$-algebra with carrier 4 and operations defined as follows: put $[(\bot/\top)^{\mathfrak{K}_{4,01}} \triangleq (0/3)$ as well as] both $(a(\wedge|\vee)^{\mathfrak{K}_{4[,01]}}b) \triangleq (\min|\max)(a,b)$ and $\sim^{\mathfrak{K}_{4[,01]}} a \triangleq (3-a)$, for all $a, b \in 4$. Then, we have the surjective $e \in \hom(\mathfrak{K}_{4[,01]}, \mathfrak{B}_{4[,01]-\mathsf{n}})$ defined by:

$$\begin{aligned} e(0) &\triangleq \mathsf{f}, \\ e(1) &\triangleq \mathsf{b}, \\ e(2) &\triangleq \mathsf{b}, \\ e(3) &\triangleq \mathsf{t}. \end{aligned}$$

[7]This clarifies the peculiarity of semantics of rather derivable rules than merely derivable axioms, studied here as well as in [27], [30], [32] and [39].

Therefore, there is some $f \in \hom(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{K}_{4[,01]})$ such that $g = (f \circ e)$. Moreover, we have the $e' \in \hom(\mathfrak{K}_{4[,01]}, \mathfrak{B}_{4[,01]-\mathsf{nb}})$ defined by:

$$\begin{array}{rcl} e'(0) & \triangleq & \mathsf{f}, \\ e'(1) & \triangleq & \mathsf{f}, \\ e'(2) & \triangleq & \mathsf{t}, \\ e'(3) & \triangleq & \mathsf{t}, \end{array}$$

in which case $h \triangleq (f \circ e') \in \hom(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{B}_{4[,01]-\mathsf{nb}})$. Assume $g[\Gamma] \subseteq \{\mathsf{t}\}$. Then, $f[\Gamma] \subseteq \{3\}$, in which case $h[\Gamma] \subseteq \{\mathsf{t}\}$, and so there is some $\psi \in \Delta$ such that $h(\psi) = \mathsf{t}$. Thus, $f(\psi) \in \{2, 3\}$, in which case $g(\psi) \in \{\mathsf{t}, \mathsf{b}\}$, and so $(\Gamma \rightarrowtail \Delta) \in (g^{-1}[\langle\{\mathsf{t}\}, \{\mathsf{t}, \mathsf{b}\}\rangle]\ddagger)$. In this way, Theorem 4.2 completes the argument. ∎

This demonstrates the power of the algebraic technique provided by the conception of fuzzy matrix as well as of the generic semantic approach elaborated here.

4.2. Three multiple-conclusion Gentzen-style axiomatizations of FDE.

4.2.1. *A multiplicative calculus.* By $\mathbb{LB}_{[01]}^{[\{\neg\}]}$ we denote the two-side propositional $\Sigma_{[01]}^{[\{\neg\}]}$-sequent calculus constituted by basic structural rules, Sharings, Cuts, the above rules $(\wedge)$ and $(\vee)$ [as well as both Left $(\bot)$ and Right $(\top)$] collectively with the following ones:

$$\begin{array}{ccc}
 & \text{Left} & \text{Right} \\
(\sim\vee) & \dfrac{(\Gamma \cup \{\sim\phi, \sim\psi\}) \rightarrowtail \Delta}{(\Gamma \cup \{\sim(\phi \vee \psi)\}) \rightarrowtail \Delta} & \dfrac{\{\Gamma \rightarrowtail (\Delta \cup \{\sim\phi\}); \Gamma \rightarrowtail (\Delta \cup \{\sim\psi\})\}}{\Gamma \rightarrowtail (\Delta \cup \{\sim(\phi \wedge \psi)\})} \\
(\sim\wedge) & \dfrac{\{(\Gamma \cup \{\sim\phi\}) \rightarrowtail \Delta; (\Gamma \cup \{\sim\psi\}) \rightarrowtail \Delta\}}{(\Gamma \cup \{\sim(\phi \wedge \psi)\}) \rightarrowtail \Delta} & \dfrac{(\Gamma \rightarrowtail (\Delta \cup \{\sim\phi, \sim\psi\}))}{\Gamma \rightarrowtail (\Delta \cup \{\sim(\phi \vee \psi)\})} \\
(\sim\sim) & \dfrac{(\Gamma \cup \{\phi\}) \rightarrowtail \Delta}{(\Gamma \cup \{\sim\sim\phi\}) \rightarrowtail \Delta} & \dfrac{\Gamma \rightarrowtail (\Delta \cup \{\phi\})}{\Gamma \rightarrowtail (\Delta \cup \{\sim\sim\phi\})} \\
[(\sim\top) & \sim\top \rightarrowtail & \\
(\sim\bot) & & \rightarrowtail \sim\bot \\
\{(\langle\sim\rangle\neg) & \dfrac{\Gamma \rightarrowtail (\Delta \cup \{\langle\sim\rangle\phi\})}{(\Gamma \cup \{\langle\sim\rangle\neg\phi\}) \rightarrowtail \Delta} & \dfrac{(\Gamma \cup \{\langle\sim\rangle\phi\}) \rightarrowtail \Delta}{\Gamma \rightarrowtail (\Delta \cup \{\langle\sim\rangle\neg\phi\})}\}]
\end{array}$$

where $\phi, \psi \in \mathrm{Fm}_{\Sigma_{[01]}^{[\{\neg\}]}}$ and $\Gamma, \Delta \in \wp_\omega(\mathrm{Fm}_{\Sigma_{[01]}^{[\{\neg\}]}})$. These are the calculi introduced and studied in [24].

Lemma 4.6. $\mathbb{LB}_{[01]}^{[\{\neg\}]}$ *is multiplicative.*

Proof. Note that $\mathbb{G} \triangleq (\mathbb{LB}_{[01]}^{[\{\neg\}]} \setminus (\mathbb{B}^2_{\Sigma_{[01]}^{[\{\neg\}]}} \cup \mathbb{S}^2_{\Sigma_{[01]}^{[\{\neg\}]}} \cup \mathbb{C}^2_{\Sigma_{[01]}^{[\{\neg\}]}}))$ is schematic. Moreover, for any $(\Gamma \to \Phi) \in \mathbb{G}$ and all $\Psi \in \mathrm{Seq}^2_{\Sigma_{[01]}^{[\{\neg\}]}}$, it holds that $((\Gamma \uplus \Psi) \to (\Phi \uplus \Psi)) \in \mathbb{G}$. Hence, $\mathbb{G}$ is multiplicative. Then, Proposition 3.1 completes the argument. ∎

Let $D \triangleq \{\mathsf{t}, \mathsf{b}\}$ and $\mathcal{B}'_{4[,01(,\neg)]} \triangleq \langle \mathfrak{B}_{4[,01]}^{[(\neg)]}, \langle D, D\rangle\rangle$.

Theorem 4.7 (Completeness Theorem; cf. [24]). *The consequence of* $\mathbb{LB}_{[01]}^{[\{\neg\}]}$ *is left-dual defined by* $\mathcal{B}'_{4[,01\{,\neg\}]}$.

Proof. First, $(\sim^{\mathfrak{B}_{4[,01]}^{[\{\neg\}]}} \sim^{\mathfrak{B}_{4[,01]}^{[\{\neg\}]}} [\{| \sim^{\mathfrak{B}^{\neg}_{4,01}} \neg^{\mathfrak{B}^{\neg}_{4,01}}\}])a = (a[\{|\neg^{\mathfrak{B}^{\neg}_{4,01}} \sim^{\mathfrak{B}^{\neg}_{4,01}} a\}])$, for all $a \in 2^2$. Therefore, $(\sim\sim)$ are true in $\mathsf{S}_\ddagger(\mathcal{B}'_{4[,01\{,\neg\}]})$. Next, D is a prime filter of the Boolean lattice $\mathfrak{B}_{4[,01]}^{[\{\neg\}]} \upharpoonright \{\wedge, \vee\}$, while $\sim^{\mathfrak{B}_{4[,01]}^{[\{\neg\}]}} [D] = \{\mathsf{f}, \mathsf{b}\}$ is a prime ideal of it [whereas $(\bot_{\mathfrak{B}_{4,01}^{\{\neg\}}} / \top^{\mathfrak{B}_{4,01}^{\{\neg\}}}) \notin / \in D$ is the zero/unit of it $\{\neg^{\mathfrak{B}^{\neg}_{4,01}} a$ being the complement of any $a \in 2^2\}$]. Hence, all rules in $\mathbb{LB}_{[01]}^{[\{\neg\}]}$ other than basic structural ones as well as both Cuts and Sharings are true in $\mathsf{S}_{\ddagger}(\mathcal{B}'_{4[,01\{,\neg\}]})$. Thus, by Corollary 3.12, we get $\vdash_{\mathbb{LB}^{[\{\neg\}]}_{[01]}} \subseteq \vdash^{\mathrm{ld}}_{\mathcal{B}'_{4[,01\{,\neg\}]}}$.

Conversely, by Lemma 4.6 and Corollary 3.11, $\vdash_{\mathbb{LB}^{[\{\neg\}]}_{[01]}}$ is left-dual defined by a left-dual reflexive transitive crisp-decomposable 2-side propositional $\Sigma^{[\{\neg\}]}_{[01]}$-sequent semantics S. Let $\Sigma' \triangleq \Sigma^{[\{\neg\}]}_{[01]}$. Consider any $v \in \mathsf{S}$, in which case there is some $C_0 \subseteq \mathrm{Fm}_{\Sigma'}$ such that $v = (\langle C_0, C_0\rangle\ddagger)$, and so, by $(\sim\sim)$, the right equalities in (18) and (19) hold, when setting $C_1 \triangleq (\mathrm{Fm}_{\Sigma'} \setminus {\sim}^{-1}[C_0])$. Likewise, by $(\langle\sim\rangle\wedge)$ and $(\langle\sim\rangle\vee)$ [as well as both Left/Right $(\bot/\top)$ and $(\sim(\bot|\top))$], both (20) and (21) [as well both (22) and (23)] hold too. [{Finally, by $(\langle\sim\rangle\neg)$, any $\varphi \in \mathrm{Fm}_{\Sigma'}$ belongs to $C_{0\langle+1\rangle}$ iff $\neg\varphi$ does not so.}] Then, taking Claim 4.3 into account, we conclude that $h \triangleq (\prod_{i\in 2} \chi^{C_i}_{\mathrm{Fm}_{\Sigma'}}) \in \hom(\mathfrak{Fm}_{\Sigma'}, \mathfrak{B}^{[\{\neg\}]}_{4[,01]})$ is such that $C_0 = h^{-1}[D]$. Thus, $v \in \mathsf{S}_{\ddagger}(\mathcal{B}'_{4[,01\{,\neg\}]})$, in which case $\mathsf{S} \subseteq \mathsf{S}_{\ddagger}(\mathcal{B}'_{4[,01\{,\neg\}]})$, and so $\vdash_{\mathbb{LB}^{[\{\neg\}]}_{[01]}} \supseteq \vdash^{\mathrm{ld}}_{\mathcal{B}'_{4[,01\{,\neg\}]}}$, as required. ■

This provides a much more immediate and transparent insight into the Completeness Theorem for $\mathbb{LB}^{[\{\neg\}]}_{[01]}$, originally proved in [24], thus once more demonstrating the power and usefulness of the generic elaboration presented here.

Since D is a prime filter of $\mathfrak{B}^{[\{\neg\}]}_{[01]} \upharpoonright \{\wedge, \vee\}$, taking the truth of distributive lattice and De Morgan identities (cf. [25]) in $\mathfrak{B}^{[\{\neg\}]}_{[01]}$ into account, by Theorem 4.7, we also get:

Corollary 4.8. *Let* $\Gamma \in \wp_{\omega}(\mathrm{Fm}_{\Sigma^{[\{\neg\}]}_{[01]}})$ *and* $\bar{\varphi} \in \mathrm{Fm}^{+}_{\Sigma^{[\{\neg\}]}_{[01]}}$. *Then,*

$$\begin{aligned}
(\Gamma \rightarrowtail (\mathrm{img}\,\bar{\varphi})) &\dashv\vdash_{\mathbb{LB}^{[\{\neg\}]}_{[01]}} (\Gamma \rightarrowtail (\vee\bar{\varphi})),\\
((\mathrm{img}\,\bar{\varphi}) \rightarrowtail \Gamma) &\dashv\vdash_{\mathbb{LB}^{[\{\neg\}]}_{[01]}} ((\wedge\bar{\varphi}) \rightarrowtail \Gamma),\\
(\sim(\vee\bar{\varphi}) \rightarrowtail \Gamma) &\dashv\vdash_{\mathbb{LB}^{[\{\neg\}]}_{[01]}} (\sim[\mathrm{img}\,\bar{\varphi}] \rightarrowtail \Gamma),\\
(\Gamma \rightarrowtail\sim(\wedge\bar{\varphi})) &\dashv\vdash_{\mathbb{LB}^{[\{\neg\}]}_{[01]}} (\Gamma \rightarrowtail\sim[\mathrm{img}\,\bar{\varphi}]).
\end{aligned}$$

4.2.2. *Non-multiplicative calculi.* Let $\mathbb{LBWC}^{[/(\neg)]}_{[\sim/01]}$ be the two-side propositional $\Sigma^{[/(\neg)]}_{[/01]}$-sequent calculus constituted by basic structural rules, Sharings, Cuts, $(\wedge|\vee)$ and $(\sim\sim)$ [as well as "$(\sim)$ with $\Gamma = \Delta = \varnothing$"/"Left|Right $(\bot|\top)$ (together with $(\neg)$)"] collectively with the following rules:

$$\text{Weak Contraposition} \qquad \frac{\phi \rightarrowtail \psi}{\sim\psi \rightarrowtail\sim\phi},$$

where $\phi, \psi \in \mathrm{Fm}_{\Sigma^{[/(\neg)]}_{[/01]}}$, in which case $(\sim\langle\wedge|\vee[|\bot|\top(|\neg)]\rangle)$ [and $(\sim)$ with $\Gamma = \Delta = \varnothing$] are derivable in it, and so, by Corollary 4.8, its consequence, being thus an extension of {the consequence of} $\mathbb{LB}^{[(\neg)]}_{[01]}$, contains the following rules:

$$\frac{\Gamma \rightarrowtail \Delta}{\sim[\Delta] \rightarrowtail\sim[\Gamma]}, \tag{24}$$

where $\Gamma, \Delta \in \wp_{(\omega\setminus 1)[\cup 1]}(\mathrm{Fm}_{\Sigma^{[/(\neg)]}_{[/01]}})$, those with $\Gamma, \Delta \in \wp_{(2\setminus 1)[\cup 1]}(\mathrm{Fm}_{\Sigma^{[/(\neg)]}_{[/01]}})$ being {either trivial, when $\Gamma = \Delta = \varnothing$, or} in it.

Now, let $\mathcal{B}^{\leqslant}_{4[,01(,)]}$ be the 2-side fuzzy $\Sigma^{[(\neg)]}_{[01]}$-matrix with underlying algebra $\mathfrak{B}^{[(\neg)]}_{4[,01]}$ and 2-fold fuzzy set having grading lattice $\mathfrak{B}_4 \upharpoonright \{\wedge, \vee\}$ and diagonal membership functions.

Theorem 4.9. *Let* $\Xi \subseteq \wp_{(\omega\setminus 1)[\cup 1]}(\mathrm{Fm}_{\Sigma^{[/(\neg)]}_{[/01]}})^2 \ni \Phi$. *Then,* $(\Xi \vdash_{\mathbb{LBWC}^{[/(\neg)]}_{[\sim/01]}} \Phi) \iff (\Xi \vdash^{\mathrm{ld}}_{\mathcal{B}^{\leqslant}_{4[/,01(,\neg)]}} \Phi)$. *[In particular, the consequence of* $\mathbb{LBWC}^{/(\neg)}_{\sim/01}$ *is left-dual defined by* $\mathcal{B}^{\leqslant}_{4/4,01(,\neg)}$*.]*

Proof. The "only-if" part is by the immediate fact that $\mathbb{LBWC}^{[/(\neg)]}_{[\sim/01]} \subseteq \vdash^{\mathrm{ld}}_{\mathcal{B}^{\leqslant}_{4[/,01(,\neg)]}} \Psi$, in view of the truth of distributive bounded lattice and De Morgan identities (cf. [25]) in $\mathfrak{B}_{4,01}$.

Conversely, assume $(\Gamma \to \Phi) \notin \vdash_{\mathbb{LBWC}^{[/(\neg)]}_{[\sim/01]}} \supseteq \vdash_{\mathbb{LB}^{[/(\neg)]}_{[\sim/01]}}$, in which case, by (24), $((\Gamma \cup \Gamma') \to \Phi) \notin \vdash_{\mathbb{LB}^{[/(\neg)]}_{[\sim/01]}}$, where $\Gamma' \triangleq \{\sim [\Delta] \rightarrowtail \sim [\Theta] \mid (\Theta \rightarrowtail \Delta) \in \Gamma\}$, and so, by Theorem 4.7, there is some $h \in \hom(\mathfrak{Fm}_{\Sigma^{[(\neg)]}_{[01]}}, \mathfrak{B}^{[(\neg)]}_{4[,01]})$ such that $(\Gamma \cup \Gamma') \subseteq (h^{-1}[D]\ddagger) \not\ni \Phi$. Then, since D and $(2^2 \setminus (\sim^{\mathfrak{B}_4})^{-1}[D]) = \{\mathsf{t}, \mathsf{n}\}$ are exactly all prime filters of the distributive lattice $\mathfrak{B}_4 \restriction \{\wedge, \vee\}$, by the Prime Ideal Theorem for distributive lattices, we have $(\Phi|\Gamma) \notin | \subseteq (h^{-1}[\vec{F}^{\mathcal{B}^{\leqslant}_{4[,01(,\neg)]}}]\ddagger) \in \mathsf{S}_{\ddagger}(\mathcal{B}^{\leqslant}_{4[,01(,\neg)]})$. ∎

Since D is a prime filter of the lattice $\mathfrak{B}_4 \restriction \{\wedge, \vee\}$, in which case axioms of $\vdash^{\mathrm{ld}}_{\mathcal{B}^{\leqslant}_{4[,01(,\neg)]}}$ are those of $\vdash^{\mathrm{ld}}_{\mathcal{B}'_{4[,01(,\neg)]}}$, by Theorems 4.7 and 4.9, we, first, get:

Corollary 4.10. *Derivable axioms of* $\mathbb{LB}^{/(\neg)}_{/01}$ *are exactly those of* $\mathbb{LBWC}^{/(\neg)}_{[\sim]/01}$. *In particular,* $\mathbb{LB}(\mathbb{WC})_{([\sim])}$ *is a Gentzen-style axiomatization of FDE.*

Nevertheless, such is not, generally speaking, the case for proper rules. More precisely, we have:

Corollary 4.11. $((\{p_0, p_2\} \rightarrowtail p_1) \to (\{\sim p_1, p_2\} \rightarrowtail \sim p_0)) \notin \vdash_{\mathbb{LBWC}^{[(\neg)]}_{[01]}}$, *in which case* $((p_0 \rightarrowtail p_1) \to (\sim p_1 \rightarrowtail \sim p_0)) \in \mathbb{LBWC}^{[(\neg)]}_{[01]}$ *is not derivable in* $\mathbb{LB}^{[(\neg)]}_{[01]}$, *and so* $\vdash_{\mathbb{LBWC}^{[(\neg)]}_{[01]}} \supsetneq \vdash_{\mathbb{LB}^{[(\neg)]}_{[01]}}$. *In particular,* $\mathbb{LBWC}^{[/(\neg)]}_{[\sim/01]}$ *is not multiplicative. Likewise, neither of* $(\sim)$ *with* $\Gamma = \Delta = \varnothing$ *belonging to* $\mathbb{LBWC}_{\sim}$ *is derivable in* $\mathbb{LBWC}$, *in which case the consequence of the former is a distinct extension of the one of the latter, and so the consequences of* $\mathbb{LBWC}_{\sim}$, $\mathbb{LBWC}$ *and* $\mathbb{LB}$ *are distinct from one another.*

Proof. First, the left|right side of any premise of any rule of $\mathbb{LBWC}$ is empty, whenever that of its conclusion is so,[8] in which case no Right|Left $(\sim)$ rule with empty $\Delta|\Gamma$ is derivable in $\mathbb{LBWC}$, and so its consequence is distinct from the one of $\mathbb{LBWC}_{\sim}$. Next, let $h \in \hom(\mathfrak{Fm}_{\Sigma^{[(\neg)]}_{[01]}}, \mathfrak{B}^{[(\neg)]}_{4[,01]})$ extend $\{\langle p_0, \mathsf{b}\rangle, \langle p_1, \mathsf{f}\rangle\} \cup ((V_\omega \setminus \{p_0, p_1\}) \times \{\mathsf{n}\})$, in which case $(\{p_0, p_2\} \rightarrowtail p_1) \in (h^{-1}[\vec{F}^{\mathcal{B}^{\leqslant}_{4[,01(,\neg)]}}]\ddagger) \not\ni (\{\sim p_1, p_2\} \rightarrowtail \sim p_0)$, and so Lemma 4.6 and Theorem 4.9 complete the argument. ∎

Thus, it is $\mathbb{LBWC}_{\sim}$ that appears the right deductive fragment of $\mathbb{LBWC}^{(\neg)}_{01}$.

The non-optional version of the above completeness theorem collectively with [25] provides a much more immediate and transparent insight into the algebraic completeness theorem for the non-empty-both-of-sequent-sides fragment of $\mathbb{LBWC}$ originally obtained in [23, 24] with using rather esoteric advanced algebraically-logical tools, not at all applicable to proving the optional left-side version of the above theorem just because of absence of interpretation of propositional Σ-sequents with empty either side by means of (sets of) Σ-equations. And what is more, the

[8]This is why the non-empty-either-/both-of-sequent-sides fragments of $\mathbb{LBWC}$ /"explicitly studied in [23, 24]" is well-defined.

optional right-side version of the above theorem (collectively with [28]) provides a new as well as much more immediate and transparent insight into the algebraic completeness theorem for $\mathbb{LBWC}_{01}^{(\neg)}$ originally obtained in [24] with using rather esoteric advanced algebraically-logical tools. These points once more demonstrate the power and usefulness of the generic elaboration presented here. And what is more, the elaboration of this subsection is equally applicable to the multiplicative calculus introduced and studied in [20] as well as its non-multiplicative extension introduced and studied in [23].

5. Conclusions

Thus, we have completely implemented the research program announced in Section 1. We should like to highlight that, as for two-side sequent calculi, the principal advance of the present study with regard to [27] consists in not merely extending the scopes of fuzzy semantics of derivable rules of sequent calculi by eliminating Sharings and/or Cuts but mainly clarifying how (more specifically, where from) fuzziness arises when dealing with non-multiplicative calculi. Namely, this is just because direct products of non-one-element families of crisp sets are always fuzzy.

Perhaps, it is equally noteworthy that a one more substantial advance of the present study with regard to [27] consists in Booleanity and complete distributivity (i.e., atomic Booleanity; cf. [55]) of complete grading lattices of fuzzy valuations/matrices constituting fuzzy /matrix semantics of /propositional sequent calculi with *merely basic* structural rules, the lattice completeness, not highlighted therein *explicitly*, being due to argumentation of main results obtained therein.

As for possible directions of further work, it would be interesting and valuable to develop an algebraic study of fuzzy matrices, advanced applications of which have been demonstrated by a one more algebraic proof of the Cut Elimination Theorem for Gentzen's calculus [10] (cf. Corollary 4.5).[9] In addition, it would be especially valuable to explore single-conclusion calculi as to such a fuzzy semantics. The problem is that, as opposed to multiple-conclusion calculi, single-conclusion ones do not possess universal disjunction (like $\uplus$ for former ones), so this point is far from being straightforward. These advanced issues go far beyond the scopes of the present paper constituting merely foundations of the topic involved and are going to be discussed elsewhere.

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[9]Recall that the first algebraic argumentation of this theorem, following a quite different and much more complicated method, has been due to [54]. In this way, the present paper provides a new and much more transparent insight into the issue. It appears that Cut Elimination (at least, in the propositional case and under presence of inverse logical rules) is just due to certain homomorphisms from the four-element Kleene lattice onto the three- and two-element ones.

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