An Exotic 4-sphere

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Abstract

It has not been known whether or not there are any exotic 4-spheres: such an exotic 4-sphere would be a counterexample to the smooth generalized Poincaré conjecture in dimension 4. Some plausible candidates are given by Gluck twists, but many cases over the years were ruled out as possible counterexamples. In the paper the resulting solution to the last generalized Poincaré conjecture is presented by giving a precise construction of a discrete exotic 4-sphere (Berkovich analytic spaces and the Richter-Gebert's Universality theorem help).

1 Introduction

I can not avoid the following quote of Isaac Newton:

"If I have seen further, it is by standing on the shoulders of giants."

By late sixties it was understood that the detection whether a given topological manifold of dimension > 4 is smoothable can be decided by homotopic theoretic methods. However, it is now understood that the smoothability issue for 4-manifolds goes beyond homotopy theory and the elegant machinery of homotopy theory [1] breaks in this dimension.

For piecewise linear manifolds, the generalized Poincaré conjecture is true except possibly in dimension 4, where the answer has been unknown, and **equivalent** to the smooth case (the last is due to Robion Kirby and Laurence Siebenmann).

A PL structure on an manifold is equivalent to a combinatorial triangulation. However, every (simplicial) triangulation of a 4-dimensional manifold is combinatorial (it requires the 3-dimensional Poincaré conjecture, which has been proven by Grigori Perelman, since a simplicial complex is a piecewise linear manifold iff the link of each simplex is a piecewise linear manifold equivalent to the standard piecewise linear sphere, for which the piecewise structure induced by the triangulation of the sphere as the boundary of a simplex).

Hence, an **exotic triangulation** for a 4-sphere (which is provided in the paper) refutes the smooth 4-dimensional Poincaré conjecture. Note that Pachner moves [2] (analogous, in some way, to the well-known Reidemeister moves of knot theory; the move is the replacement of k n-simplices by (n - k + 2) n-simplices gluing along boundaries) are a way to manipulate triangulations. The theorem of Udo Pachner states that whenever two triangulated manifolds are PL-equivalent, there is a **finite** sequence of Pachner moves transforming both into another.

As a side note, in the introduction of Bruce Blackadar and Joachim Cuntz's paper [3] it is stated that a piecewise linear structure on a compact topological n-manifold may be regarded as a suitable choice of generators of the commutative unital C*-algebra of complex valued continuous function on it.

Next, the result of Frank Quinn states that every topological 4-manifold is smoothable away from an arbitrary point, i.e. this means that all corners with the possible exception of one can be can smoothed out (all PL 4-manifolds are simple branched covers of the 4-sphere [4]). The Simon Donaldson's work produced examples of 4-manifolds for which this one last corner cannot be removed. Consequently, the above with a stereographic projection and examples of large exotic $\mathbb{R}^4$ (an exotic $\mathbb{R}^4$ is called large if it cannot be smoothly embedded as an open subset of the standard $\mathbb{R}^4$) give insight why the smooth 4-dimensional Poincaré conjecture is not true.

If M is a smooth manifold homeomorphic to $\mathbb{R}^4$ and there is a smoothly embedded $S^3 \hookrightarrow M$ so that the unbounded component of its complement is diffeomorphic to $(0, \infty) \times S^3 \cong \mathbb{R}^4 \setminus \{0\}$, then a new chart around infinity can be simply added to create a smooth structure on the 1-point compactification (if $\mathbb{R}^4$ was exotic, so would be this new 4-sphere). Moreover, for an exotic 4-sphere S and for all $x \in S$ the complement $S \setminus \{x\}$ is diffeomorphic to an exotic $\mathbb{R}^4$ (every oriented embedding of the closed 4-ball in S is isotopic, i.e. any two cylindrical end smoothings are isotopic; it due to Jean Cerf and Richard Palais).

To conclude, many potential exotic 4-spheres have been constructed. Historically, one of the most promising families is the family of Cappell-Shaneson. One reason why they appeared encouraging is that two Cappell-Shaneson homotopy 4-spheres are known to double cover exotic 4-dimensional real projective spaces.

To date, the only known way to verify an exotic manifold is indeed exotic comes from gauge theory and related invariants. However, at present these invariants are unable to give information in the case of a 4-manifold with trivial second homology. Due to this, much of the work done on Cappell-Shaneson homotopy 4-spheres has been in attempting to show that they are not exotic (using Kirby calculus; somewhat analogous to a cell complex).

Hence, note that the graph of triangulations / flip-graph (the set of all triangulations under adjacency by Pachner moves (Pachner moves are also called bistellar flips)) is known to be connected for the vertex sets of Cartesian products of two simplices if one of them has dimension **at most three** (Francisco Santos and Gaku Liu).

2 Why 4?

It has been established that there is no difference between piecewise linear and smooth manifolds in dimension < 7 (Robion Kirby and Laurence Siebenmann). On the other hand, **every PL n-sphere (any n) becomes polytopal** (the convex hull of finitely many points / a partially ordered set, the elements and maximal totally ordered subsets of which are called faces and flags respectively, such that certain properties are satisfied) after finitely many derived subdivisions [5]. Notice that polytopality can be decided by a finite algorithm.

Lemma. *Two piecewise linear structures on a n-sphere are unequivalent if and only if the related n-polytopes are unequivalent.*

Proof. It follows from [5], since two triangulations are related by a finite sequence of Pachner moves/bistellar subdivisions if and only if they define PL-homeomorphic piecewise linear manifolds [2].

Note that a subdivision is a triangulation if every cell is a simplex. A stellar subdivision of the link of a face F in a simplicial complex is obtained by removing F and adding a new vertex v along with all simplices formed from v, a proper subface of F, and a face in the link of F. A bistellar operation on a simplicial n-sphere is a certain combination of a stellar subdivision and inverse stellar subdivision at the same site. A derived subdivision is obtained by stellarly subdividing at all faces in order of decreasing dimension. In other words, a derived subdivision doesn't violate equivalence coming from bistellar subdivisions.◀

The Hauptvermutung states that two triangulations always admit a common subdivision. The assumption was proven for manifolds of dimension ≤ 3 and for differentiable manifolds but it was disproved in general (John Milnor; via the combinatorial invariant of Reidemeister Torsion).

The Whitehead theorem states that every weak homotopy equivalence between CW-complexes (every simplicial complex is a CW-complex; different rules for what kinds of gluings one is allowed to use) is a homotopy equivalence. Thus, we oppositely come **from handle decompositions to CW-complexes.**

Corollary. *Two piecewise linear structures on a compact 4-manifold are unequivalent if and only if the sets of the related 4-polytopes are not congruent by subdivisions.*

Proof. Once again, a PL structure on an manifold is equivalent to a combinatorial triangulation. However, every (simplicial) triangulation of a 4-dimensional manifold is combinatorial (a simplicial complex is a piecewise linear manifold if and only if the link of each simplex is a piecewise linear manifold equivalent to the standard piecewise linear sphere, for which the piecewise structure induced by the triangulation of the sphere as the boundary of a simplex).

Thus, consider sets of 4-simplexes, since the 1, 2, 3-dimensional world is not exotic.◀

The answer is the following:

i. As extension of the Whitney's theorem n-polytopes are determined by their (n-2)-skeletons (the k-skeleton is the union of the simplices of dimensions $\leq k$). However, simplicial n-polytopes are determined by their $\lfloor d/2 \rfloor$-skeletons (Micha Perles). So, there is a crucial difference: for 4-polytopes the reduction to combinatorial graph problem doesn't exist.

ii. The Richter-Gebert's Universality theorem states that the realization space of a 4-polytope can be arbitraly wild, i.e. for every basic primary semialgebraic set defined over $\mathbb{Z}$ there is a 4-polytope whose realization space is stably equivalent to the semialgebraic set. However, all combinatorial types of 3-polytopes can be realized with rational vertices. Moreover, the shape of any facet of any 3-polytope can be preassigned (David Barnette and Branko Grünbaum). In addition, it is interesting that every graph is a spanning subgraph (contains all the vertices of the original graph) of the graph of a 4-polytope.

iii. For infinitely many different non-polytopal 5-spheres, every subcomplex on fewer vertices can be extended to the boundary of a polytope (Bernd Sturmfels).

Remark 1. *The secondary polytope is a construction of Andrei Zelevinsky, Israel Gelfand and Mikhail Kapranov, where its vertices correspond to the so-called regular triangulations.*

It is important to keep in mind that the secondary polytope and the flip graph are not combinatorial invariants, but depend on its metrical properties.

Remark 2. *Schlegel diagrams are useful for visualization of 4-polytopes.*

Remark 3. *The graph (1-skeleton) of every n-polytope is n-connected (every pair of vertices is connected by n internally disjoint paths or, equivalently, the removal of any n-1 vertices leaves a connected graph with at least two vertices; due to Michel Balinski).*

Remark 4. *Notice that if M is a n-manifold which is a homology sphere, then the double suspension is homeomorphic to a (n + 2)-sphere (James Cannon and Robert Edwards). It gives an example of a triangulation of a topological sphere that is not piecewise linear.*

Remark 5. *How many simplicial spheres are there, of a given dimension and number of vertices? What are possible numbers of faces of different dimensions of a simplicial sphere (the g-conjecture, formulated by Peter McMullen)?*

The upper bound theorem on spheres (Richard Stanley) implies that the number of combinatorially different n-spheres with k vertices is in $exp(O(k^{\lceil \frac{n}{2} \rceil} log(k)))$*. On the other hand, Gil Kalai showed how to construct* $exp(O(k^{\lfloor \frac{n}{2} \rfloor}))$ *of them (cyclic polytopes). And he proved that, in fact, "most" simplicial spheres are non-polytopal.*

The g-conjecture was proven by Karim Adiprasito in the more general context of rational homology spheres [6] *(the hard Lefschetz theorem for toric varieties).*

3 Main result

From the first two Sections we know that any differentiable 4-manifold can be triangulated (piecewise-linearly). Moreover, to disprove smooth 4-dimensional Poincaré conjecture it is enough to exhibit a discrete/triangulated exotic 4-sphere.

By standard piecewise linear structure of a n-sphere, the piecewise linear structure induced by the triangulation of a n-sphere as the boundary of a (n+1)-simplex is meant. Note that in case $n = 4$ it defines the 4-dimensional cross-polytope also goes by the name hexadecachoron or 16-cell. It is one of the six convex regular 4-polytopes.

Remark 6. *The n-cross-polytope is a nonfacet for* $n \geq 4$*. A polytope is equifacetted if all its facets are of the same combinatorial type. A n-polytope is facet-forming if it is the combinatorial type of the facets of some equifacetted (n + 1)-polytope; otherwise it is a nonfacet.*

Remark 7. *The k-skeleton of a n-polytope, where* $1 \leq k \leq n - 1$*, contains a subdivision of the k-skeleton of the n-simplex (Branko Grünbaum; it is 5-cell in n=4 case).*

Convex 4-polytopes can be cut and unfolded as nets in 3-space. Since n-polytopes are determined by their (n-2)-skeletons, the 2-skeleton (union of 2-faces (flat **surfaces**)) features a 4-polytope. Then standard piecewise linear structure of a 4-sphere defines a **tiling** by convex regular polygons.

Remark 8. *In 2-dimensions, there are only two types of Pachner moves (do not forget about the inverse). First, there is a Pachner move which one can perform on any triangle and which replaces it by three triangles given by coning off from the center of the original triangle. This move is called a 1-3 move, since 1 triangle is replaced by 3 triangles.*

There is also a 2-2 move which replaces two distinct triangles glued along a common edge. The two triangles glued along the common edge form a quadrilateral, and a 2-2 move replaces one diagonal edge of this quadrilateral by the other. For this reason, a 2-2 move is also commonly called a diagonal exchange. Note that the quadrilateral could have other edges being identified (and Pachner moves could be performed to exchange them too).

The idea in the paper finds common ground with p-adic numbers and beyond:

0) Lemma and Corollary in the previous Section.

1) A **Berkovich space** is a version of an analytic space or Serre's GAGA over a non-Archimedean field, e.g. p-adic field (Grothendieck topologies and rigid analytic geometry of John Tate, tropical varieties should be mentioned) [12].

Non-Archimedean analytic techniques are powerful for studying algebraic varieties over the complex numbers, since non-Archimedean fields such as $\mathbb{C}_p$ and the completion of $\mathbb{C}\{\{t\}\}$ are isomorphic to $\mathbb{C}$ as abstract fields (any two uncountable algebraically closed fields of the same cardinality and characteristic are isomorphic). Thus, for a variety over $\mathbb{C}_p$ with a certain collection of algebraic properties there exists a variety over $\mathbb{C}$ with the same collection of properties.

Note that Vladimir Berkovich proved that **analytifications** of smooth varieties are **locally contractible** and have **the homotopy type of a finite simplicial complex** (skeletons of formal models), see works of Matthew Baker.

The theory of perfectoid spaces (a category of some geometry objects over a field of characteristic 0 and characteristic p is equivalent) by Peter Scholze is also notable.

2) The another important brick comes from Oliver Knill.

A **finite simplicial complex can be recovered** both from the connection graph (in which the vertices are the elements in the complex and where two sets are connected if they intersect) and the **Barycentric refinement** graph (in which two vertices are connected if and only if one set is contained in the other).

For any simplicial complex the connection graph and the Barycentric refinement graph have the same automorphism group. Moreover, **the Cartesian product**, **the Stanley-Reisner product** as well as **the strong product** have the same automorphism group, which is the product group of the automorphism groups. In addition, expected homotopy property is preserved in all cases.

3) The **Rado graph** is a notable object for the purpose, since it can be characterised by its universality and homogeneity. The graph has countably many vertexes, and it is universal in the sense that any graph with countably many vertexes is isomorphic to an induced subgraph of it. Moreover, any isomorphism between finite induced subgraphs can be extended to the whole (homogeneity). These properties (as usual) determine the Rado graph uniquely up to isomorphism. Moreover, removing any finite set of its vertexes and edges produces a graph isomorphic to the whole Rado graph (**robustness**).

4) Samuel Eilenberg and Norman Steenrod have shown that any compact space may be expressed as an **inverse limit** in the category of topological spaces of a diagram of simplicial complexes. Moreover, any paracompact space may be expressed as a limit of a diagram of nerves.

In addition, Sibe Mardešić has associated with every inverse system of compact CW-complexes and every simplicial complex with geometric realization a certain resolution, which consists of spaces having the homotopy type of polyhedra. It was shown that this construction is functorial. It was generalized by Nikica Uglešic and Branko Cervar: every topologically complete space **embeds as a deformation retract** in a topologically complete space which is the limit of a polyhedral inverse system with surjective and simplicial bonding mappings and the corresponding homotopy category and its full subcategory are equivalent (the same also holds for paracompact spaces, Lindelöf spaces, countably compact spaces, paracompact (σ-compact) locally compact spaces, compact Hausdorff spaces).

Consider the graph of all triangulations of a 4-sphere. There is an edge from one triangulation to another if and only if there exists a subdivision which makes the first triangulation be the second.

Lemma from the previous Section allows to define a **valuation** as the minimal number of subdivisions needed for making a triangulation be polytopal. There do exist (Cartesian) product and sum (as the disjoint union) for simplicial complexes, but it is is possible to use operations defined on graphs due to 2). Note that subtraction comes in manner of Grothendieck rings and K-theory, i.e. the graph of all triangulations of a 4-sphere possesses the structure of a commutative ring. To add, one can get the field in the manner of p-adic numbers, but is there geometry in that?

Furthermore, the graph of triangulations / flip-graph (the set of all triangulations under adjacency by Pachner moves) and the constructed valuation lead to a Banach norm. The last gives the spectrum, i.e. the set of all non-zero bounded multiplicative **seminorms** (the spectrum is provided with the weakest topology).

Theorem. *The smooth 4-dimensional Poincaré conjecture is wrong.*

Proof. A finite n-polytope is the geometric realization of some finite simplicial complex (note that apeirotopes are polytopes with infinitely many cells), which in turn is a compact smooth n-manifold. Notice that there are only **countably** many finite simplicial complexes, so it follows that there are only countably many compact differentiable 4-manifolds.

According to Lemma, Corollary and the Richter-Gebert's Universality theorem (i.e. ii.) from the previous Section, there exists an exotic compact 4-manifold. Thus, Corollary from the previous Section gives existence of an exotic 4-sphere as an exotic 4-simplex.◀

Corollary. *There are uncountably many exotic 4-spheres.*

Proof. It also follows from the Richter-Gebert's Universality theorem (ii. in the previous Section), see the just proved Theorem.◀

Corollary. *The smooth 1, 2, 3-dimensional Poincaré conjecture is right.*

Proof. With the observations (i. and ii. in the previous Section) it is a trivial task now.◀

What are combinatorial explanations for 5-spheres and 6-spheres to have only one smooth structure? See iii. from the previous Section as a clue.

An enormous number of different **invariants** coming from the structure of a simplicial complex appears. Matroids (geometrical lattices), the Minkowski sum, the Dehn invariant and etc show up. What are relations with Donaldson and Seiberg–Witten theories? The guess is in **weighted simplicial complexes** (each simplex of the simplicial complex is associated to a real number called weight).

A sum over simplicial geometries is a sum over the different ways the simplices can be joined together with an integral over their edge lengths (the information about topology is contained in the rules by which the simplices are joined together; metric is provided by an assignment of edge lengths to the simplices and a flat metric to their interiors; curvature is concentrated on the two-dimensional triangles in which they intersect).

After reading the next Section the question raises: **What has left?**

4 Remarks

Remark 9. ***The reconstruction conjecture in graph theory*** *asks: are graphs uniquely determined by their subgraphs? It has been shown by Béla Bollobás* [8] *that almost all graphs are reconstructible, i.e. the probability that a randomly chosen graph of n vertices is not reconstructible goes to 0 as n tends to infinity.*

Check also the paper [9] *of Richard Stanley about switching-reconstructible graphs.*

Moreover, are almost all graphs uniquely determined by the spectrum of their adjacency matrix? It might seem plausible for all graphs to be spectrally determined, but this is false. The problem resonates with the famous "Can you hear the shape of a drum?".

In addition, can every simple connected graph of n vertices be decomposed into at most $\frac{1}{2}(n+1)$ paths?

Remark 10. *The Vizing's conjecture concerns a relation between the domination number and the Cartesian product of graphs* [10].

Remark 11. *Does every countable graph have an unfriendly partition into two parts? An unfriendly partition is a partition of the vertices of the graph into disjoint subsets, so that every vertex has at least as many neighbors in other sets as it has in its own set. It is a generalization of the concept of a maximum cut for finite graphs. However, Saharon Shelah and Eric Milner showed that an unfriendly partition into three subsets always exists (even for uncountable graphs)* [11].

Remark 12. *A link to the first exotic sphere (which were constructed by John Milnor* [12] *in dimension-7 using Quaternionic Hopf fibration) reminds mystery. The classification of exotic spheres by Michel Kervaire and John Milnor showed that the oriented exotic 7-spheres are the non-trivial elements of a cyclic group of order 28 under the operation of connected sum (it has known that there exists only one smooth structure for a 3-sphere).*

Remark 13. *The paper* [13] *examines end sums and cancellation of (possibly infinite) collections of 0- and 1-handles at infinity.*

Remark 14. *Discrete geometry is a relatively new development, the books* [14][15] *are recommended.*

Remark 15. *The Borel conjecture, which claims that any homotopy equivalence between aspherical closed manifolds (it is path connected and all its higher homotopy groups vanish) is homotopic to a homeomorphism, is an important open problem in topology.*

This is not true in general: there are homotopy equivalent lens spaces which are not homeomorphic. On the other hand, there is the Mostow rigidity theorem. And the Borel conjecture is a topological reformulation of Mostow rigidity, weakening it from hyperbolic manifolds (connected with constant sectional curvature -1) to aspherical manifolds and similarly weakening an isometry to a homeomorphism.

The Borel conjecture also finds connections with Ramsey theory and with the of all self-homeomorphism of $\mathbb{R}^n$ equipped with the compact open topology.

Thomas Farrell and Lowell Jones have made deep contributions to the field.

The Borel conjecture implies the Novikov conjecture (concerns the homotopy invariance of certain polynomials in the Pontryagin classes of a manifold, arising from the fundamental group) for the special case. The Novikov conjecture is equivalent to the rational injectivity of the assembly map in L-theory (the K-theory of quadratic forms). The Borel conjecture on the rigidity of aspherical manifolds is equivalent to the assembly map being an isomorphism.

Remark 16. *Higher-dimensional minimal models (also called Mori's program; it tries to get rid of negative rays for coming to a nef divisor) in birational geometry are feasible, provided one is careful* ***about the types of singularities which occur*** *(the names of Vyacheslav Shokurov and Caucher Birkar should be mentioned here).*

The problem of termination of log flips in higher dimensions (≥ 4) remains the subject of active research. Notice that any ***algebraic variety*** *over $\mathbb{R}$ or $\mathbb{C}$* ***admits a triangulation*** *(Kyle Hofmann).*

Remark 17. *Calabi-Yau manifolds are complex (simply connected) manifolds that are generalizations of K3 surfaces in any number of complex dimensions. They are defined as compact Kähler manifolds with a vanishing first real Chern class (or equivalently, with trivial canonical bundle). It admits a Ricci-flat metric via Calabi's conjecture = Yau's theorem. Ricci flatness finds applications in theoretical physics: the extra dimensions of space-time are sometimes conjectured to take the form of a 6-dimensional Calabi–Yau manifold, which led to the idea of mirror symmetry. Moreover, they are in some sense on* ***the boundary*** *between the better understood class of Fano type varieties and the varieties of general type, for which there is no hope of general understanding.*

One of the most important tools in the investigation of such complex manifolds is the feature that ***their singularities are connected with the structure of Lie algebras****.*

In three complex dimensions ($6 < 7$), classification of the possible Calabi–Yau manifolds is an open problem. However, the Wall's theorem says that a Calabi-Yau threefold is topologically characterized by its Hodge numbers, second Chern class and intersection ring.

Over 473 million toric embeddings of Calabi-Yau threefolds were constructed (Victor Batyrev in the paper [16] *found a very elegant description in terms of* ***reflexive polyhedrons****; the toric set-up (weighted complex projective spaces) is an algebraic property that can be analyzed in terms of combinatorial algebra), with over 30,000 distinct Hodge diamonds (Maximilian Kreuzer and Harald Skarke). So, it yields at least 2590 distinct diffeomorphism classes.*

Hence, as observed later, many examples of pairs of topologically distinct Calabi-Yau threefolds can be connected. For instance, Yujiro Kawamata proved that any two birational smooth Calabi-Yau manifolds (in any complex dimension) can be connected by a sequence of flops.

It has been conjectured by Miles Reid that the number of topological types of Calabi–Yau threefolds is infinite, and that ***they can all be transformed continuously*** *(conifolds) one into another (as Riemann surfaces can).*

A novel way to classify Calabi-Yau threefolds by systematically studying their infinite volume limits is presented in [17].

5 Last remark

See the other paper [1] **of the author about the homotopy groups of spheres.** It utilizes correspondence between the i-th homotopy group of (r+1)-sphere and the i-th homotopy group of the wedge sum of i (r+1)-spheres based on the Hilton's theorem (the homotopy groups of such wedge sums consolidate all information about homotopy groups of spheres).

Due to the classical Hopf fibration it is known that $\pi_i(S^2)$ and $\pi_i(S^3)$ are isomorphic whenever i is at least 3. So, **the case $r = 2$ is special** (check also the Hilton's theorem for the i-th homotopy group of the wedge sum of i (2+1)-spheres to observe this; 2 plays the role) and it is possible to calculate all homotopy groups of (2+1)-sphere.

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