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Abstract

In this study, we consider a single objective fuzzy portfolio optimization with flexible goal and constraints, in which the Sharpe ratio is chosen as the goal and the portfolio's mean and variance are included in the constraints. Although this problem has much significance in finance, it is difficult to solve because of the nonconvexity of the objective function. Based on fuzzy theory and flexible optimization, the fuzzy portfolio problem is transformed to the crisp form which is proved to be a semistrictly quasiconvex programming problem for any decreasing membership functions. This property of the equivalent problem is the basis to solve the main problem efficiently by available convex programming algorithms. The computational experiments with the SP500 data set is reported to show the performance of the proposed model.

Keywords: Fuzzy portfolio selection, Flexible optimization, Soft constraints, Sharpe ratio, Semistrictly quasiconvex programming.

1 Introduction

In financial technology, portfolio selection is a significant process, specially in business investments. It is the process of assigning a predetermined amount of money to various assets in order to construct a well-balanced portfolio. This process helps determine a mix of securities from a large number of possibilities. Its goal is to help investors get the most out of their money. In recent years, portfolio selection has gotten much attention from many individuals and institutions.

Mean-Variance model or Markowitz model is popular and used widely in modern portfolio theory, where portfolio selection is constructed as a bi-objective optimization problem for minimizing risk of portfolio as well as maximising expected portfolio profit. As usual, this problem is transformed to a single-objective programming problem, such as minimizing risk of portfolio for a chosen level of anticipated return or maximising expected portfolio profit for a chosen level of risk. Because both the risk and profit objective functions are convex, the transformed problem is a convex programming problem.

In practice, the actual data, however, we can see that a number of variables impact the stock market, resulting in frequent price changes as well as our desire to find the best answer to the difficult situation (see [1]). As a consequence, we can see that tackling the problem of fuzzy portfolio selection is a great idea. Numerous more results that applied fuzzy theory to portfolio problems were presented recently (see [2–4]).

Besides mean and variance, some other measures are also used to evaluate portfolios (see [2, 5, 6]). This research considers a variant Markowitz model by adding an objective function called Sharpe Ratio (SR) (also see [7]). SR is a prominent risk-adjusted return performance metric that indicates the ratio of the expectant profit to the standard deviation of the portfolio. SR has much applied and economic significance, but it has some disadvantages when solving the programming problem with the objective SR because of its non-convexity. Therefore, recently there only some works related to this index, such as Sharpe Ratio - VaR Ratio model with fuzzy coefficients solved using genetic algorithm (see [4]) ,the "fuzzy Sharpe Ratio" and new risk measure with fuzzy random variables in the fuzzy modeling environment (see [2]). These works used the heuristic approach such as genetic algorithms for solving the programming problem. This approach only helps find the local optimal solutions and cannot guarantee the algorithmic convergence to global optimal solutions. Moreover, for the computational performance aspect, it may have to execute a huge computational load to generate a big enough population to find a good enough approximated solution.

In this research, we show the semistrictly quasiconvex property of Sharpe Ratio function. By utilizing the properties of semistrictly quasiconvex functions , the fuzzy portfolio selection problem using SR is transformed to a semistrictly quasiconvex programming, which can solve globally and efficiently by available convex programming algorithms (also see [8, 9]). The globally convergence of the proposed procedure is guaranteed. Last but not least, we also prove that the equivalent crisp problem is still a semistrictly quasiconvex programming even when the membership functions are arbitrary decreasing functions, while the previous works only consider some specific forms of membership functions (see [10]). This helps one get more flexible, easily and suitable options for fuzzification in the modelling process.

The remainder of the paper is arranged as follows. In section 2, we present the portfolio selection problem which uses the Sharpe Ratio index. Section 3 introduces the fuzzy portfolio selection issue and the solution approach. The findings of the computational experiments are shown in Section 4. In the final portion, there are a few conclusions.

2 Portfolio selection problem

Markowitz's portfolio theory is founded on two key assumptions: 1) investors are wary about risks and want the highest possible expected profit, and 2) investors pick their portfolios based on desirable return and variance of return.

Recall the random vector $\boldsymbol{R} = (R_1, R_2, \ldots, R_n)^T \in \mathbb{R}^n$ denotes random returns of the n assets. Suppose that $p(\boldsymbol{R})$ is the probability distribution of $\boldsymbol{R}$. Calling mean vector of $\boldsymbol{R}$ is $\boldsymbol{L} = (L_1, \ldots, L_n)^T$ and covariance matrix of $\boldsymbol{R}$ is $\boldsymbol{Q} = (\sigma_{ij})_{n\times n}$, where σ_{jj}^2 is variance of R_j and σ_{ij}^2 is correlation coefficient between R_i and R_j, $i, j = 1, \ldots, n$. With $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ is a portfolio, note that $\sum_{j=1}^n x_j = 1$ and $x_j \geq 0$ for all $j = 1, \ldots, n$. We have the expected return $\mathcal{E}(\boldsymbol{x}) = E[\boldsymbol{R}^T\boldsymbol{x}] = \sum_{j=1}^n L_j x_j$, and variance of profit $\mathcal{V}(\boldsymbol{x}) = Var(\boldsymbol{R}^T\boldsymbol{x}) = \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_j x_i = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$.

The expected return is used to represent the investment in the future, whilst the variance of return is used to estimate the risk of the investment. As a result, the return is calculated by the projected profit or the average of the profits, while the profit variance throughout the whole portfolio is used to estimated the risk. Investors are best served by two objectives: maximum lucrative and minimum the risk with portfolio value bound constraints.

Set $M = \{\boldsymbol{x} \in \mathbb{R}_+^n \mid \sum_{j=1}^n x_j = 1\}$ where $\mathbb{R}_+^n$ is nonnegative orthant of real n-dimentional space. Then, the portfolio optimization problem is therefore presented as follows.

$$\begin{array}{ll} \max & \mathcal{E}(\boldsymbol{x}) = E[\boldsymbol{R}^T\boldsymbol{x}] = \sum_{j=1}^n L_j x_j \\ \min & \mathcal{V}(\boldsymbol{x}) = Var(\boldsymbol{R}^T\boldsymbol{x}) = \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_j x_i = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}, \\ s.t. & \boldsymbol{x} \in M. \end{array} \tag{MV}$$

The Sharpe Ratio function, on the other hand, is the aim of our article - an index that measures a portfolio's risk-adjusted returns with the formulation as

$$\mathcal{S}(\boldsymbol{x}) = \frac{\mathcal{E}(\boldsymbol{x}) - r_f}{\sqrt{\mathcal{V}(\boldsymbol{x})}}$$

where r_f represents the risk free rate. The higher the SR value, the higher the portfolio return in comparison to the risk the decision-maker is willing to take.

In this research, we propose the single-objective deterministic portfolio optimization model as

$$\begin{aligned} \min\ & f(\boldsymbol{x}) = -\mathcal{S}(\boldsymbol{x}) = -\frac{\mathcal{E}(\boldsymbol{x}) - r_f}{\sqrt{\mathcal{V}(\boldsymbol{x})}} \\ s.t.\ & \mathcal{E}(\boldsymbol{x}) = E[\boldsymbol{R}^T\boldsymbol{x}] = \textstyle\sum_{j=1}^n L_j x_j \geq \alpha, \\ & \mathcal{V}(\boldsymbol{x}) = Var(\boldsymbol{R}^T\boldsymbol{x}) = \textstyle\sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_j x_i = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} \leq \beta, \\ & \boldsymbol{x} \in M. \end{aligned} \tag{MVS1}$$

The basic theory of semistrictly quasiconvex programming will be presented later in Section 3.3. In this section, there are some propositions related to semistrictly quasiconvexity that can be inferred from the proposed model.

Recall that if $\varphi_1(\boldsymbol{x})$ is concave and $\varphi_2(\boldsymbol{x})$ is convex defined on $C \subset \mathbb{R}^n$ satisfied $\varphi_1(\boldsymbol{x}) \geq 0, \varphi_2(\boldsymbol{x}) > 0$, then $\phi = \dfrac{\varphi_1}{\varphi_2}$ is semistrictly quasiconcave function on C (see [11], table 5.4, page 165). Therefore, we have the propositions below.

Proposition 1 *The Sharpe Ratio function $\mathcal{S}(\boldsymbol{x})$ is semistrictly quasiconcave on $\mathbb{R}^n$.*

Proposition 2 *Problem (MVS1) is a semistrictly quasiconvex programming.*

In actuality, there are some inaccuracies in the parameters, as well as some errors in estimating the expected profit and the risk of the portfolio. If we continue to work on the original objective function, we may not be able to find a satisfactory optimal solution. Furthermore, fuzzicating constraints expands the feasible set, increasing the likelihood of obtaining a satisfied solution with a better optimal value. Therefore, considering the Problem (MVS1) in the fuzzy context is reasonable. In the next section, we will go over some fundamental concepts in fuzzy optimization, the procedure for converting Problem (MVS1) to a fuzzy form, and how to solve the fuzzy problem.

3 Fuzzy portfolio problem

3.1 Fuzzy optimization

In this article, we use the fuzzy optimization model with fuzzy components in which the variables and parameters are crisp numbers, which is also called flexible optimization (see [12]). There are two types of fuzzy components. The first component is associated with the objective function ($\widetilde{\min}, \widetilde{\max}$), while the second component is a fuzzy relation ($\preceq, \simeq, \succeq$). Without loss the generality, a fuzzy optimization problem with fuzzy components can be formulated as

$$\begin{aligned} \widetilde{\min}\ & f(\boldsymbol{c}, \boldsymbol{x}) \\ s.t.\ & z_i(\boldsymbol{x}) \preceq 0, \quad i = 1, \ldots, k, \\ & \boldsymbol{x} \in X, \end{aligned} \tag{P_f}$$

where $X \subset \mathbb{R}^n$ is a set that is not empty. The symbol "$\widetilde{\min}$" and "$\preceq$" stand for a fuzzy version of "minimize" and "$\leq$", respectively, indicating "the objective function should be reduced as much as feasible" and "the constraints should be possibly well acceptable."

In the following, we review the fundamentals of fuzzy theory.

Definition 1 (Fuzzy set ([10])) Given an universal set X. To define a fuzzy set $\tilde{A}$ of X, a function μ is used.

$$\mu_{\tilde{A}} : X \longrightarrow [0, 1]$$

It is called membership function, which sets a real number $\mu_{\tilde{A}}$ to each $x \in X$ in order to represent the level of membership of x in $\tilde{A}$.

In general case, a membership function has the form as

$$\mu_i(z_i(x)) = \begin{cases} 0 & \text{if } z_i(x) \geq z_i^0, \\ d_i(x) & \text{if } z_i^0 \geq z_i(x) \geq z_i^1, \\ 1 & \text{if } z_i(x) \leq z_i^1, \end{cases}$$

where $d_i(x)$ is a monotonic non-increasing function with respect to z_i, z_i^0 and z_i^1 represents the value of z_i such that the grade of $\mu_i(z_i(x))$ is 0 or 1. Therefore, membership functions are decreasing functions. Membership function for objective function and constraints present the grade that the objective function is minimized and the constraints are satisfied, respectively.

In many other previous studies, researchers propose solving methods for only some particular types of membership functions, but our approach can be used for arbitrary strictly decreasing functions.

3.2 Fuzzy portfolio problem

In this paper, we adopt vagueness and transform Problem (MVS1) to its fuzzy version as

$$\begin{array}{ll} \widetilde{\min} & f(\boldsymbol{x}) = -\mathcal{S}(\boldsymbol{x}) = -\dfrac{\mathcal{E}(\boldsymbol{x}) - r_f}{\sqrt{\mathcal{V}(\boldsymbol{x})}} \\ s.t. & \mathcal{E}^*(\boldsymbol{x}) = -\mathcal{E}(\boldsymbol{x}) = -\sum_{j=1}^{n} L_j x_j \preceq -\alpha, \\ & \mathcal{V}(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} \preceq \beta, \\ & \boldsymbol{x} \in M. \end{array} \tag{MVS2}$$

By applying arbitrary strictly decreasing membership functions $\mu_i, i = 0, 1, 2$ (*) and following the fuzzy decision of Bellman and Zadeh, Problem (MVS2) is transformed to Problem (FP1).

$$\begin{array}{ll} \max \min & \left\{ \mu_0(f(\boldsymbol{x})), \mu_1(\mathcal{E}^*(\boldsymbol{x})), \mu_2(\mathcal{V}(\boldsymbol{x})) \right\} \\ s.t. & \boldsymbol{x} \in M. \end{array} \tag{FP1}$$

Because $f(\boldsymbol{x})$ is not convex, Problem (FP1) is a nonconvex programming problem. However, it belongs to the class of semistrictly quasiconvex programming problems. The reason will be presented in Subsection 3.3.

3.3 Semistrictly quasiconvex programing

Definition 2 (Semistrictly quasiconvex function (see [11])) Let $X \subset \mathbb{R}^n$ be a convex set, f is defined on X. For all $\boldsymbol{x}^1 \in X, \boldsymbol{x}^2 \in X$, $0 < \lambda < 1$, if

$$f(\boldsymbol{x}^1) > f(\boldsymbol{x}^2) \quad \text{implies that} \quad f(\lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) < f(\boldsymbol{x}^1),$$

then f is semistrictly quasiconvex on X.

Definition 3 (Semistrictly quasiconvex programming problem) Given a convex set $X \subset \mathbb{R}^n$, $X \neq \varnothing$. Then the formulation of a semistrictly quasiconvex programming problem is

$$\begin{array}{ll} \min & f(\boldsymbol{x}) \\ s.t. & \boldsymbol{x} \in X, \end{array} \tag{SQP}$$

where $f(\boldsymbol{x})$ is semistrictly quasiconvex on X.

Proposition 3 *Any local minimum of* (SQP) *is a global minimum (see [11]). Therefore,* (SQP) *can be solved by proper convex programming algorithms.*

Let's return to Problem (FP1). By Proposition 2, $f(\boldsymbol{x})$ is a semistrictly quasiconvex function. In addition, $\mathcal{E}^*(\boldsymbol{x})$ is linear and $\mathcal{S}(\boldsymbol{x})$ is convex because of the positive symmetric definite matrix $\mathbf{Q}$. So we have $f(\boldsymbol{x})$, $\mathcal{E}^*(\boldsymbol{x})$ and $\mathcal{S}(\boldsymbol{x})$ are semistrictly quasiconvex functions (**).

Set operator $\eta_i(y) = 1 - \mu_i(y)$, then Problem (FP1) is equivalent to

$$\begin{array}{l} \min \ g(\boldsymbol{x}) = \max\left\{\eta_0(f(\boldsymbol{x})), \eta_1(\mathcal{E}^*(\boldsymbol{x})), \eta_2(\mathcal{V}(\boldsymbol{x}))\right\} \\ s.t. \ \ \boldsymbol{x} \in M. \end{array} \tag{FP2}$$

By assumption (*), $\eta_i(y)$, $i = 0, \ldots, 2$ are strictly increasing functions. Combining with (**), then, $\eta_0(f(\boldsymbol{x})), \eta_1(\mathcal{E}^*(\boldsymbol{x})), \eta_2(\mathcal{S}(\boldsymbol{x}))$ are semistrictly quasiconvex functions (see [11], Proposition 5.1, page 154). After that, we have Proposition 4, which can be proven by using Definition 2. And finally, all the reasons stated lead to Proposition 5.

Proposition 4 *The objective function* $g(\boldsymbol{x})$ *of Problem (FP2) is semistrictly quasiconvex on* $\mathbb{R}^n$.

Proposition 5 *Problem (FP2) is a semistrictly quasiconvex programming problem and is solved efficiently by using available convex programming algorithms.*

The proposed method simplifies the computation process significantly. This approach is different from the other approach of the previous articles which use genetic algorithms.

4 Computational Experiment

This section uses two examples to describe the performance of our proposal in the application of fuzzy portfolio optimization.

The first example uses stock prices from 1/23/2015 to 6/12/2017 of five stocks with symbols such as ULTA, MLM, NFLX, AMZN and NVDA. The expected profits and covariance matrix of five assets are given in Table 1.

Table 1: Data of the Example 1

Stock	$\boldsymbol{L}$	$\boldsymbol{Q}$				
ULTA	0.15672	4.41513	1.12491	2.31042	1.44398	1.39347
MLM	0.15874	1.12491	4.07482	1.96306	1.28708	1.53560
NFLX	0.20462	2.31042	1.96306	9.13912	2.33831	1.98378
AMZN	0.21693	1.44398	1.28708	2.33831	4.43169	1.67068
NVDA	0.34876	1.39347	1.53560	1.98378	1.67068	5.31435

The deterministic problem is (MVS1) model with $\alpha = 2.5$, $\beta = 0.25$. Then, the membership functions applied for objective function and constraints are piecewise-linear membership functions which are strictly decreasing functions on the range of $f(\boldsymbol{x}), \mathcal{E}^*(\boldsymbol{x}), \mathcal{V}(\boldsymbol{x})$.

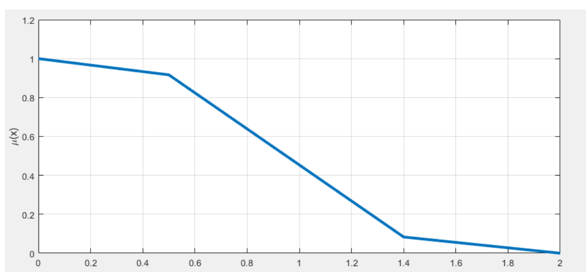

Fig. 1: Piecewise-linear membership function

The fuzzy portfolio problem is solved using Matlab's convex programming tool, and the results are presented in Table 2. From Table 2, the optimal value for the fuzzy portfolio problem is 1% greater than the optimal value for the deterministic problem. Flexible optimization that expands the feasible set can look to produce a better objective function value than the initial problem's optimal value.

The second example uses five stocks ticker symbols such as ALB, AVGO, CHTR, ULTA and MLM. And Table 3 displays the data's expected profits and covariance matrix of stocks cost from 2/2/2015 to 1/25/2017.

To analyze the discovered solutions, we conduct the following: after determining the optimal solution $\overline{\boldsymbol{x}}$ of the portfolio selection problem, we utilize it

Table 2: Value of (MVS1) and (MVS2) with $\alpha = 0.25$ and $\beta = 2.5$

Function	$\mathcal{S}(\boldsymbol{x})$	$\mathcal{E}(\boldsymbol{x})$	$\mathcal{V}(\boldsymbol{x})$
Deterministic ($\overline{\boldsymbol{x}}$)	0.14649	0.25363	2.49999
Fuzzy ($\boldsymbol{x}^*$)	0.14795	0.26151	2.62045

Table 3: Data of the second example

Stock	$\boldsymbol{L}$	$\boldsymbol{Q}$				
ALB	0.15540	3.72379	1.53106	1.11161	1.05120	1.35014
AVGO	0.15881	1.53106	5.29711	1.08504	1.69505	1.57479
CHTR	0.16339	1.11161	1.08504	4.00362	1.03543	1.20725
ULTA	0.16504	1.05120	1.69505	1.03543	4.39397	1.11849
MLM	0.17814	1.35014	1.574787	1.20725	1.11849	4.06780

to compute the optimal solution $\boldsymbol{x}^*$ of the fuzzy portfolio problem by applying piecewise-linear membership functions. Then, we calculate the average of Sharpe ratio, mean, and variance values in the next 100 days with each solution $\overline{\boldsymbol{x}}$ and $\boldsymbol{x}^*$. Table 4 displays the results. The SR value of the fuzzy problem is not as good as it is for the deterministic problem, as seen in Table 4. However, When applied to data over the next 100 days, the average of $S(\boldsymbol{x}^*)$ is 1.12% better than $S(\overline{\boldsymbol{x}})$, demonstrating the second meaning of fuzzy.

Table 4: Result of the second example

Function	$\mathcal{S}(\overline{\boldsymbol{x}})$	$\mathcal{S}(\boldsymbol{x}^*)$	$\mathcal{E}(\overline{\boldsymbol{x}})$	$\mathcal{E}(\boldsymbol{x}^*)$	$\mathcal{V}(\overline{\boldsymbol{x}})$	$\mathcal{V}(\boldsymbol{x}^*)$
500 days	0.10576	0.10521	0.16574	0.16426	1.84720	1.82834
Mean of 100 data sets	0.10276	0.10391	0.15802	0.15855	1.75269	1.72776

5 Conclusion

In conclusion, this work proposes a portfolio model, in which we optimize the Sharpe ratio and control the value of return and risk, and consider the model in the fuzzy environment. We have stated that the deterministic problem is a (SQP), and proposed a method, which can use just arbitrary strictly decreasing membership functions, to convert it to the flexible version that then was proven to be a (SQP). As the result, the fuzzy portfolio problem can be addressed by just using convex programming methods, which takes less computing effort than genetic algorithms in previous workarounds. Because of its advantages in practical experiments compared to other methods and the appearance of semistrictly quasiconvex functions in many problems in the real world, this approach may be used in a variety of different mathematical models that contain semistrictly quasiconvex functions.

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