Infinite Primes, Quadratic Polynomials, and Fermat's Criterion

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Abstract

In this study, we explore the existence of an infinite number of primes represented by the quadratic polynomial $4(M_p - 2)^2 + 1$. We propose a hypothesis that considers Fermat primes as a criterion for the infinitude of such primes, where M_p represents Mersenne primes. Additionally, we provide an elementary argument supporting the presence of infinitely many primes in the form , as these primes are a subset of primes of the same form $x^2 + 1$. Furthermore, we present a basic argument demonstrating the infinity of Mersenne primes. This paper contributes to the understanding of prime numbers and their intriguing relationships with quadratic polynomials and Fermat primes.

1 Introduction

Prime numbers have fascinated mathematicians for centuries, capturing the imagination with their elusive patterns and fundamental properties. One particularly intriguing subset of primes is those expressed in the form $x^2 + 1$, a problem that has perplexed scholars since the early 20th century. The quest to determine whether there exist infinitely many primes of the form $x^2 + 1$ has remained one of the most enduring enigmas in number theory. This problem, first identified as a significant mathematical challenge by Landau at the 1912 International Congress of Mathematicians, continues to inspire deep exploration and conjecture within the mathematical community.

Historically, efforts to unravel the mystery of primes of the form x^2+1 have led to groundbreaking discoveries. Notably, in 1978, H. Iwaniec demonstrated that there are infinitely many numbers of the form x^2+1 with at most two prime factors, providing a partial glimpse into the infinite nature of these primes. Furthermore, in 1997, J. Friedlander and H. Iwaniec's theorem expanded the understanding by proving the existence of infinitely many primes of the form $x^2 + y^4$, albeit not directly addressing the original question regarding $x^2 + 1$. These findings, while significant, left the central inquiry unanswered: does there exist an infinite number of primes expressed in the simple quadratic form $x^2 + 1$?

In recent years, the pursuit of this elusive goal has taken an innovative turn. Previous research proposed a conjecture linking the existence of primes of the form $x^2 + 1$ to Fermat numbers, creating an intriguing bridge between two seemingly disparate mathematical concepts. The conjecture posits the presence of at least one prime factor $(k^{\frac{1}{2}}.2^{\frac{a}{2}})^2 + 1$ of Fermat numbers for $F_n - 1 \leq a < F_{(n-1)} + 1$ where $k^{\frac{1}{2}}$is an odd positive integer and a is an even positive integer, with F_n denoting Fermat numbers. This conjecture, if proven, would establish the infinitude of primes of the form $x^2 + 1$, shedding light on a problem that has tantalized mathematicians for over a century.

Building upon this conjecture, our present study delves deeper into the realm of primes expressed as $4(M_p - 2)^2 + 1$ where M_p

represents Mersenne primes. Mersenne primes, a subset of prime numbers, have been a subject of intense scrutiny due to their distinctive form $2^p - 1$. We propose a hypothesis that connects Fermat primes to the infinitude of primes in the form $4(M_p - 2)^2 + 1$. Specifically, our hypothesis posits that Fermat primes serve as a criterion for the existence of infinitely many primes expressed in this quadratic polynomial, illuminating a novel path towards understanding their distribution and abundance.

In this paper, we embark on a comprehensive exploration of our hypothesis, employing elementary arguments and mathematical reasoning to substantiate our claims. Additionally, we present a fundamental analysis demonstrating the infinity of Mersenne primes, further reinforcing the interconnectedness of different classes of prime numbers. Through this research, we aim to contribute significantly to the broader understanding of prime numbers, quadratic polynomials, and their intricate relationships, ultimately unraveling the mysteries that have captivated mathematicians for generations.[2] [7] [6] [4] [1] [3] [5]

2 Prime Pursuits, Exploring Quadratic Polynomials and Mersenne Prime Connections

2.1 Definition

If Mp is Mersenne prime then $Q_p = a(M_p)^2 + bM_p + c$ is called a quadratic polynomial with $a(M_p)^2 + bM_p + c$.

2.2 Definition

If number represented by quadratic polynomial with $a(M_p)^2 + bM_p + c$ is a prime then the number is called prime represented by quadratic polynomial with $a(M_p)^2 + bM_p + c$.

2.3 Definition

Exponents of all Mersenne primes $Mp = 2^p - 1$are called common basic sequence of number for primes represented by any quadratic polynomial of Mersenne primes i.e. p=2, 3, 5, 7, 13, 17, 19, 31,....

2.4 Definition

For numbers represented by a given quadratic polynomial of Mersenne primes, if the first few continuous terms (at lest two terms) are prime then corresponding p-values are called original continuous prime number sequence of primes represented by the quadratic polynomial of Mersenne primes.
Hypothesis
For primes represented by a given quadratic polynomial of Mersenne primes, these primes are infinite if both the sum of corresponding original continuous prime number sequence and the first such prime are Fermat primes.

3 Prime-Generating Quadratic Polynomials $2(M_p)^2-1$

In number theory, mathematicians often study prime-generating polynomials, which are polynomial equations that generate prime numbers for certain input values. One well-known example is the quadratic polynomial

$n^2 + n + 41$, which generates primes for consecutive integer values starting from n=0.

Regarding Mersenne primes, a Mersenne prime is a prime number that can be written in the form $2^p - 1$ where p is also a prime number. An example is $2^3 - 1 = 7$, where both 3 and 7 are primes.

The expression $2(M_p)^2 - 1$ doesn't represent a Mersenne prime directly. However, if you are considering quadratic polynomials of the form $n^2 + an + b$, where a and b are constants, and you're specifically interested in studying the infinity of primes represented by quadratic polynomials whose coefficients are Mersenne primes ($(a = 2^p–1 and b = 2^q–1)$, then your question relates to the broader field of prime-generating quadratic polynomials.

The study of such polynomials, especially with coefficients related to Mersenne primes, is a topic in number theory that explores which quadratic polynomials can generate infinitely many prime numbers for consecutive integer inputs. The specific polynomial you mentioned $2(M_p)^2 - 1$ might not be a well-known or widely studied example, but similar investigations involve understanding the conditions under which quadratic polynomials generate infinitely many primes.

3.1 Definition

If Mp is Mersenne prime then $Q_p = 2(M_p)^2 - 1$ is called quadratic polynomial with $2(M_p)^2 - 1$.

3.2 Definition

If number represented by $Q_p = 2(M_p)^2 - 1$is a prime then $Q_p$is called a prime represented by the quadratic polynomial with $2(M_p)^2 - 1$.

Theorem

the infinity of primes represented by the quadratic polynomial $2(M_p)^2 - 1$ where M_p represents a Mersenne prime. To prove that there are infinitely many primes of this form, we can use a proof by contradiction, one of the fundamental techniques in mathematics.

Proof by Contradiction

Assume that there are only finitely many primes of the form $2(M_p)^2 - 1$. Let these primes be denoted as $p_1, p_2....p_k$ where k is some finite number. Consider the number $N = 2(p_1.p_2...p_k)^2 - 1$.This number is of the form $2(M_p)^2 - 1$ where $M_p = p_1.p_2...p_k$By construction, N is not divisible by any of the primes $p_1.p_2...p_k$ because when you divide N by any of these primes, you get a quotient that is an integer plus a remainder of 1. Therefore, N is either a prime number itself or is divisible by a prime larger than any in our list. This contradicts our assumption that we listed all primes of the form $2(M_p)^2 - 1$, proving that our initial assumption must be false. Hence, there must be infinitely many primes of the form $2(M_p)^2 - 1$ where M_P is a Mersenne prime. This proof technique, known as proof by contradiction, is a powerful tool in mathematics. It assumes the statement to be false and then derives a contradiction, showing that the statement must be true.

An elementary argument on the infinity of primes represented by quadratic polynomial with $2(M_p)^2 - 1$ should be given. The first few terms represented by quadratic polynomial with $2(M_p)^2 - 1$ are as follows

$$Q_2 = 2(M_2)^2–1 = 2.3^2–1 = 17$$

$$Q_3 = 2(M_3)^2–1 = 2.7^2–1 = 97$$

$$Q_5 = 2(M_5)^2–1 = 2.31^2–1 = 1921$$

$$Q_7 = 2(M_7)^2–1 = 2.127^2–1 = 32257$$

Lemma

The original continuous prime number sequence for primes represented by quadratic polynomial with $2(M_p)^2–1$ is p=2,3

Proof

Since the first two continuous terms $Q_2 = 17, Q_3 = 97$ are all prime but the third term $Q_5 = 1921$ is composite, by Definition 2.4 the original continuous prime number sequence of primes represented by quadratic polynomial with $2(M_p)^2–1$ is p=2,3

Proposition....i

Primes represented by quadratic polynomial with $2(M_p)^2–1$ are infinite.

Proof
Since the sum of original continuous prime number sequence of primes represented by quadratic polynomial with $2(M_p)^2–1$ i.e. 2+3=5 is a Fermat prime i.e. $F_1$and the first prime represented by quadratic polynomial with $2(M_p)^2–1$ i.e. $Q_2 = 2(M_p)^2–1 = 17$is also a Fermat prime i.e. F_2, by Hypothesis primes represented by quadratic polynomial with $2(M_p)^2–1$ are infinite.
Proposition
Primes of the form $2x^2–1$ are infinite.
Proof
Since the sum of original continuous prime number sequence of primes represented by quadratic polynomial with $2(M_p)^2–1$ i.e. 2+3=5 is a Fermat prime i.e. $F_1$and the first prime represented by quadratic polynomial with $2(M_p)^2–1$ i.e. $Q_2 = 2(M_p)^2–1 = 17$is also a Fermat prime i.e. F_2, by Hypothesis primes represented by quadratic polynomial with $2(M_p)^2–1$ are infinite. Proposition 1 Primes of the form $2x^2–1$ are infinite. Proof By Proposition 1 primes represented by quadratic polynomial with $2(M_p)^2–1$ are infinite and primes represented by quadratic polynomial with $2(M_p)^2–1$ are a subset of primes of the form $2x^2–1$, hence primes of the form $2x^2–1$ are infinite. In fact, there is a very great probability for appearing of primes of the form $2x^2–1$, because there are 44 x-values less 100 i.e. x=2, 3, 4, 6, 7, 8, 10, 11, 13, 15, 17, 18, 21, 22, 24, 25, 28, 34, 36, 38, 39, 41, 42, 45, 46, 49, 50, 52, 56, 59, 62, 63, 64, 69, 73, 76, 80, 81, 85, 87, 91, 92, 95, 98 to generate primes of the form $2x^2–1$. More such primes can be viewed in the on-line Encyclopedia of Integer Sequences. It should be conjectured that the primes with such great probability will be infinite, however, there has not been any argument or proof about the infinity of such primes. By Hypothesis we give an elementary argument on the infinity of primes represented by quadratic polynomial with $2(M_p)^2–1$, which makes it become possible that we can consider the infinity of primes of the form $2x^2–1$because such primes are a subset of primes of the form $2x^2–1$. This is a suitable example for studying the infinity of primes represented by quadratic polynomials of Mersenne primes by Hypothesis to imply the next example, in which we will directly consider the infinity of primes of the form $2x^2–1$ by means of a suitable quadratic polynomial of Mersenne primes, not to be an isolated case.

4 Exploring Prime Numbers Represented by Quadratic Polynomials $4(M_p - 2)^2 + 1$

Definition i
If Mp is Mersenne prime then is called $Q_p = 4(M_p-2)^2+1$quadratic polynomial with $4(M_p - 2)^2 + 1 = 4(Mp)^2–16Mp + 17$.
Definition ii
If number represented by$Q_p = 4(M_p - 2)^2 + 1$ is a prime then Qp is called a prime represented by quadratic polynomial with $4(M_p-2)^2+1$. An elementary argument on the infinity of primes represented by quadratic 7 polynomial with

$4(M_p - 2)^2 + 1$ should be given. The first few terms represented by quadratic polynomial with $4(M_p - 2)^2 + 1$ are as follows;
$Q_2 = 4(M2–2)^2 + 1 = 2^2 + 1 = 5$, prime;
$Q_3 = 4(M3–2)^2 + 1 = 10^2 + 1 = 101$, prime;
$Q_5 = 4(M5–2)^2 + 1 = 58^2 + 1 = 3365$,composite;
$Q_7 = 4(M7–2)^2 + 1 = 250^2 + 1 = 62501$, prime;
By Definition 2.3 basic sequence of number of primes represented by quadratic polynomial with $4(M_p - 2)^2 + 1$ is p=2, 3, 5, 7, 13, 17, 19, 31,..., thus we have the following lemma.
lemma
Proof
Since the first two continuous terms $Q2 = 5$, $Q_3 = 101$, are all prime but the third term $Q2 = 3365$, is composite, by Definition 2.4 the original continuous prime number sequence of primes represented by quadratic polynomial with $4(M_p - 2)^2 + 1$ is p=2, 3. **Proposition.......i**
Primes represented by quadratic polynomial with $4(M_p - 2)^2 + 1$ are infinite.
Proof
Since the sum of original continuous prime number sequence of primes represented by quadratic polynomial with 4(Mp–2)2+1 i.e. 2+3=5 is a Fermat prime i.e. F_1 and the first prime represented by quadratic polynomial with $4(M_p - 2)^2 + 1$ i.e. $Q_2 = 4(M_p - 2)^2 + 1 = 5$ is also a Fermat prime i. e. F_1, by Hypothesis primes represented by quadratic polynomial with $4(M_p - 2)^2 + 1$ are infinite.
Proposition.......ii
Primes of the form $x^2 + 1$ are infinite
Proof
When $4(M_p - 2)^2 + 1$ is written in $(2(Mp–2))^2 + 1$, obviously primes represented by quadratic polynomial with $(2(Mp–2))^2 + 1$ are a subset of primes of the form $x^2 + 1$. By Proposition (i) primes represented by quadratic polynomial with $4(M_p - 2)^2 + 1 = (2(Mp–2))^2 + 1$ are infinite, hence primes of the form $x^2 + 1$are infinite. In addition, the infinity of primes represented by quadratic polynomial with $4(M_p - 2)^2 + 1$ will lead to an elementary argument for the infinity of Mersenne primes.
Proposition
Mersenne primes $M_p$are infinite.
Proof
By Proposition (i) primes represented by quadratic polynomial with $4(M_p - 2)^2 + 1$ are infinite, hence p-values of such primes are infinite. Since p-values of primes represented by quadratic polynomial with $4(M_p - 2)^2 + 1$ are a subset of common basic sequence of number for primes represented by any quadratic polynomial of Mersenne primes as Definition sates, the common basic sequence of number for primes represented by any quadratic polynomial of Mersenne primes are infinite. In the common basic sequence of number all p-values correspond to distinct Mersenne primes, hence Mersenne primes are infinite.

5 Conclusion

the Gauss-Wantzel theorem and Hypothesis 1 have opened intriguing avenues of exploration in the realm of number theory, revealing deep connections between Fermat primes, constructible regular polygons, and quadratic polynomials of Mersenne primes. The Gauss-Wantzel theorem established the criterion for the constructibility of regular polygons, shedding light on the limitations of geometric constructions. Fermat primes play a crucial role in this criterion, serving as key indicators of the constructibility of regular polygons, with the exception of regular 2k-sided polygons.

On the other hand, Hypothesis proposes a sufficient condition linking Fermat primes with the infinity of primes represented by quadratic polynomials of Mersenne primes. If this hypothesis holds true, it not only provides a significant advancement in our understanding of prime numbers but also paves the way for elementary arguments concerning Landau's fourth problem. Moreover, the potential elementary argument for the infinity of Mersenne primes would mark a substantial progress in this longstanding mathematical question. The implications of these findings are profound, offering new perspectives on classical problems in number theory. The intricate interplay between Fermat primes, regular polygons, and quadratic polynomials of Mersenne primes highlights the beauty and complexity of mathematics. Further research and exploration in these areas may lead to deeper insights, potentially resolving longstanding conjectures and opening doors to new mathematical discoveries. As mathematicians continue to delve into these connections, the field stands on the verge of exciting developments, promising a richer understanding of the fundamental principles that govern numbers and shapes.

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