Quadratic Filters

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Abstract: **In this research paper, nonlinear digital filters based on weak superposition of binary output of rotation invariant Boolean filters (operating on the binary signals obtained by threshold decomposition of digital signal) are proposed. Also, quadratic filters based on Toeplitz weight matrix are proposed and are related to "correlation filters".**

Index Terms: rank order statistics, weak superposition, nonlinear digital filters, threshold decomposition.

I.INTRODUCTION

Dynamical systems arise in natural phenomena as well as artificial phenomena. Based on the "principle of Occam's razor", linear dynamical systems were subject to intense research activity and many interesting results were discovered. Particularly, Linear Time Invariant (LTI) systems were utilized to model phenomena arising in control theory, signal processing as well as communication theory. In digital signal processing research area, linear time invariant filters were characterized by the associated impulse response which could be of finite extent in time (i.e. FIR filters) or infinite extent in time (i.e. IIR filters). Such filters were generalized to two dimensions as well sa multiple dimensions. In one dimension, Finite Impulse Response (FIR) filters were specified on a finite window size in time. In the case of such filters, the filter output is a linear form/linear function (hyperplane) in the signal values in the window. Researchers were naturally interested in nonlinear filters whose output is a quadratic form (with the associated weight matrix) in the vector of signal values in the window. Such filters were called as "quadratic filters". Several interesting results related to such nonlinear filters were discovered and documented in [6].

The author is actively interested in research related to quadratic forms which have engineering applications. He realized that Rayleigh's theorem related to optimization of quadratic form over unit Euclidean hypersphere leads to interesting results. Researchers progressed research efforts related to rank order filters (i.e. filter outputs an order statistic such as the "maximum" of signal values in a filter window) motivated by the median filter. It readily follows that rank order filters are "permutation invariant" filters i.e the filter output remains same under permutation of signal values in a filter window. As a natural generalization, the author became interested in the study of quadratic filters (a type of nonlinear filters) that are ROTATION INVARIANT. This interest is based on the realization of rank order filters (permutation invariant filters) using Boolean filters. Specifically, Fitch et.al, proved a weak superposition property satisfied by rank order filters.It states that the output of rank order filter equals the weak superposition of binary outputs of Boolean filters operating on binary signals obtained by threshold decomposition of input digital signal (at various thresholds). In [2], the authors proved that (in case of rank order filters), the Boolean filters at each threshold are realized using Positive Boolean Symmetric Functions. It is realized that arbitrary Boolean functions (not necessarily positive Boolean functions) have interesting applications in cryptography.

From the point of view of studying Boolean functions, several interesting open research problems were formulated. For instance, counting the number of rotation invariant Boolean functions was considered an important research problem. Lakshmi et.al provided an important solution to this research problem on counting the number of rotation invariant Boolean functions. Based on this result and realization of nonlinear rank order filters (i.e. permutation invariant digital filters) using Boolean filters, the author was motivated to investigate ROTATION INVARIANT NONLINEAR DIGITAL FILTERS.

This research paper is organized as follows. In Section 2, related research literatue is reviewed. In Section 3, nonlinear digital filters based on weak superposition of binary outputs of Boolean filters (operating on binary signals obtained by threshold decomposition of digital signal) are investigated. In Section 4, rotation invariant filters are related to quadratic filters based on Toeplitz weight matrix. Such quadratic filters are related to "correlation filters" proposed in the research literature.

II. REVIEW OF RELATED RESEARCH LITERATURE

Nonlinear digital filtering received attention of researchers in signal processing. For instance, Allen Oppenheim proposed the class of homo-morphic filters. Also, motivated by robust image processing, median filters were investigated thoroughly [5]. Fitch et.al proposed the concept of threshold decomposition of a digital signal and showed that rank order filters (such as minimum filter, median filter, maximum filter) satisfy a weak superposition property based on Boolean filters [1]. It is well recognized that rank order filters are permutation invariant filters. Naturally, using the condition of rotation invariance, filters satisfying

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rotation invariance property (so called rotation invariant filter) were proposed. The author, proved that Boolean filters utilised at each threshold in rank order filter implementation (based on threshold decomposition) are associated with Positive Boolean Symmetric functions. This research paper is motivated by an effort to realize a class of rotation invariant filters based on weak superposition (in the spirit of rank order filters). Further, researchers proposed quadratic filters for interesting signal processing applications [6]. In this research paper, we propose rotation invariant quadratic digital filters and relate them to the class of correlation filters proposed in the research literature [8].

III. ROTATION INVARIANT NONLINEAR DIGITAL FILTERS BASED ON WEAK SUPERPOSITION

In order to propose an interesting class of rotation invariant nonlinear digital filters, we need the concept of threshold decomposition of a digital signal. We nterpreted threshold decomposition property from the point of view of number theory. Formally, we now explain the connection between signal processing and number theory.

Consider a digital signal in a window of length, M i.e. $\{ u[n] \}_{n=1}^{M}$ (u[n] is the signal sample value at discrete time index 'n'). Let the digital signal assume atmost 'K' values i.e. For each 'n', $u[n] \in \{ 1,2,\dots,K \}$.

Definition: Partition of an integer, L is given by

$$L = l_1 + l_2 + \cdots . + l_K,$$

where $1 \leq l_i \leq (L-1) \;\; for \;\; 1 \leq i \leq K.$

Given the above definition of partition of an integer, threshold decomposition of a digital signal in a window of size M is obtained using a "distinguished partition".

We first explain the threshold decomposition of a digital signal which assumes positive values only. We decompose u[n] into superposition of binary signals (assuming '0' or '1' values only) in the following manner

$$u[n] = \sum_{j=1}^{K} u_j[n] \;, \qquad where$$

$$u_j[\text{n}] = \begin{cases} +1 & if \;\; u[n] \geq j \\ 0 & if \;\; u[n] < j \end{cases}.$$

We specifically consider the threshold decomposition of a digital signal restricted to a window of size M. It can readily reasoned that there is no loss of generality in assuming K=M. Thus, the binary signals $\{ u_j[n] : 1 \leq j \leq M\}$ in a window of size M can be imbedded in an M x M square matrix, J.

Note: It readily follows that columns of J lead to the signal values at the corresponding time instant. Furthermore, any column of J corresponds to the partition of a signal value (at that time instant) whose summands are all '1'.

Note: Suppose, we consider a divisor 'd' of an integer 'L'. Then, we have that L = s d = d+d+…+d i.e. a partition of L, whose summands are all 'd'. We expect the weak superposition property of rank order filters to be generalized to such a partition based decomposition of signal values.

For the purposes of clarity, we now explain weak superposition property (discovered by Fitch et al) satisfied by rank order filters. The property states that kth order statistic of a signal u[n] in a window equals the weak superposition (real valued addition) of the corresponding order statistic of 'K' binary signals (obtained by threshold decomposition of the digital signal). Formally

$$P_K[\, u[1], u[2], \dots, u[M] \,\} = \sum_{j=1}^{M} P_k \left[\; u_j[1], u_j[2], \dots, u_j[M] \,\right].$$

As shown in [ ], kth order statistic of a binary signal in the window i.e. $\left[\; u_j[1], u_j[2], \dots, u_j[M] \,\right]$ equals the Positive Boolean Symmetric Function of order K in those binary variables. It readily follows that such Boolean functions are permutation invariant (in the binary variables). Thus, the condition of permutation invariance of filter operations on signal samples in a window is naturally associated with rank order filters.

In the spirit of permutation invariant Boolean functions, researchers were motivated to study "rotation invariant" Boolean functions. An interesting problem was to count the number of rotation invariant Boolean functions in M binary variables. This problem was solved using the Polya's theorem in [7]. Formally, we have the following definition:

- Definition: A Boolean function of M binary variables is said to be "rotation invariant" if $f(\, x_1, x_2, \dots, x_M \,)$ is equal in value for all the cyclic shifts of the M variables (for example $f(\, x_1, x_2, \dots, x_M \,) = f(\, x_M, x_1, \dots, x_{M-1} \,)$).

Using the above definition and the research work of Fitch et.a, we propose the class of rotation invariant nonlinear digital filters possessing the weak superposition property. Let $\{ \; x_i \}_{i=1}^{M}$ be binary variables i.e. { 0,1} valued variables. Let the thredhold decomposition of input signal $u[n] \; in \; a \; window \; of \; size \; M \; be \; provided \; by \; the \; binary$ signals $\{ \; u_j[n] \; \} \;\; for \;\; 1 \leq j \leq K.$

$$u[n] = \sum_{j=1}^{K} u_j[n] \quad for \;\; 1 \leq n \leq M.$$

Let $R[\, u_j[1], \; u_j[2], \dots., u_j[m] \,\}$ be a "rotation invariant" function of M binary variables i.e $\{ \, u_j[1], \; u_j[2], \dots., u_j[m] \, \}$. Let the nonlinear digital filter output be computed by weak superposition of output of rotation invariant Boolean filters { operating on signals in a window of size M } i.e.

$$y = R \, \{ \, u[n] \, \} \; i.e$$
$$y = R \, \{ \, u[1], u[2], \dots, u[m] \, \}$$
$$= \textstyle\sum_{j=1}^{K} R\left[\, u_j[1], \; u_j[2], \dots., u_j[m] \,\right\}.$$

Note: As in the case of rank order filters, at each threshold, the Boolean function (necessarily rotation invariant) can be a structured one. For instance, it may be constrained to be a Postive Boolean function (i.e. the class of Positive Boolean Rotation Invariant functions.

Note: The class of rotation invariant Boolean functions at each threshold can be suitably constrained (as in the case of rank order filters). The associated nonlinear digital filters based on weak superposition of the output of Boolean filters could have interesting signal processing significance.

IV. ROTATION INVARIANT QUADRATIC DIGITAL FILTERS

The concept of quadratic filters is motivated by Finite Impulse Response filter (FIR) operating on a signal in a window of finite length. More specifically, the output $y[n]\ of\ an\ FIR\ filter\ with\ impulse\ response\ \{\ h[n]\ \}_{n=1}^{M}$ (of length M), input signal { u[n] } is given by

$$y[n]\ = \sum_{j=1}^{M} h[j]\ u[n-j].$$

i.e. output is a "linear form" in the input signal sample values. Naturally, researchers innovated the concept of "Quadratic Filters". The output of such a filter is a quadratic form based on a weight matrix, B and the vector of signal sample values in a window. Let z be the output value, $\bar{x}$ *be the vector of signal sample values. Then , we have that* $z = \bar{x}^T B\ \bar{x}$.

Now, we consider a special class of quadratic filters. We have the following definition.

Definition: A quadratic filter is said to be "rotation invariant" if the output 'z' remans the same under circular shift of the components of the vector, $\bar{x}$.

The following Lemma provides a characterization of rotation invariant quadratic filters based on the nature of M x M weight matrix, B.

Lemma: A quadratic filter with the associated weight matrix B is ROTATION INVARIANT if and only if B is a symmetric Toeplitz matrix.

Proof: It readily follows that

$$z = \bar{x}^T B\ \bar{x} = \sum_{i=1}^{M} \sum_{j=1}^{M} b_{ij} x_i x_j\ .$$

Suppose the quadratic filter output 'z' is invariant under cyclic shifts of the components of the vector $\bar{x}$. Then, it necessarily follows that
$b_{ij} = b_{|i-j|}\ for\ all\ i,j.$

Hence B is necessarily a Toeplitz symmetric matrix.

Using an identical reasoning, if B is a symmetric Toeplitz matrix, the quadratic filter is also rotation invariant filter QED.

Now, we reason that the output of a rotation invariant quadratic filter with the input vector 'x' (in a filter window signal sample values) can be expressed as a cascade of "correlation filter" and an FIR filter.

$$z = \sum_{i=1}^{M} \sum_{j=1}^{N} b_{|i-j|} x_i x_j = \sum_{l=0}^{M-1} R(k)\ b_k\ ,$$ where

R(k) is the autocorrelation sequence of the sequence of components of the vector $\bar{x}$. Thus, the following figure depicts the implementation of a rotation invariant quadratic filter.

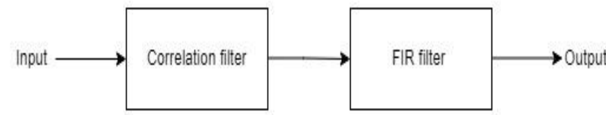

Fig.1 Implementation of rotation invariant filter

In the figure above, the correlation filter computes the autocorrelation of the signal sample values in the input vector x̄. The impulse response of the FIR filter is based on the first row components of the symmetric Toeplitz matrix B.

Example: Now, we illustrate the implementation of a rotation invariant quadratic filter with an example:

$$\bar{B} = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 3 & 1 \\ 4 & 1 & 3 \end{bmatrix},$$

$$a\ symmetric\ Toeplitz\ filter\ weight\ matrix.$$

$\bar{x} = [\ x_1\ x_2\ x_3\ ]$. We now compute the quadratic form associated with the matrix B i.e.

$$\bar{x}^T \bar{B} \bar{x} = 3\ (x_1^2 + x_2^2 + x_3^2) + 2\ x_1 x_2 + 8\ x_1 x_3 + 2\ x_2 x_3$$

= 3 R(0) + 2 R(1) + 8 R(2), where R(i) is the autocorrelation value at 'lag' 'i' .

The FIR filter coefficients are { 3, 2 (1), 2 (4) } (i.e. based on the first row elements of B).

VI. CONCLUSION:

Rotation invariant nonlinear digital filters which satisfy the weak superposition property are proposed. Also, quadratic filters based on rotation invariance property are investigated.

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